Numerical simulations of a degenerate parabolic-hyperbolic equation in a bounded domain with/without containment assumptions

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Abstract: - In this article, we focus on the evolution of numerical solutions of degenerate parabolic hyperbolic equations in one spatial dimension with boundary conditions. Under what are known as "containment" assumptions on the data, numerical simulations are presented in order to better understand these assumptions on the evolution of the solutions.

Key-Words: - Numerical simulation, Degenerate parabolic-hyperbolic equation, Dirichlet boundary condition, Zero-flux boundary condition, Robin boundary condition, "containment" assumptions, Entropy solution.

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1 Introduction

This article is dedicated to the numerical illustration of the evolution of the solution for degenerate parabolic-hyperbolic equations of the type:

$$u_t + \operatorname{div}(f(u) - \nabla\phi(u)) = 0 \text{ in } D. \tag{1}$$

Here $D = (0,T) \times I$ where I is a bounded domain of $\mathbb{R}^\ell$, $\ell \geq 1$ and $F = (0,T) \times \partial I$ with ∂I a regular boundary of I. At time $t = 0$, we add a bounded initial condition to the study of (1)

$$u(0,x) = u_0(x) \text{ in } I. \tag{2}$$

Since the domain is bounded, we are also interested in what happens at the boundary of the spatial domain. To this end, three examples of boundary conditions are examined in this article:

a) Zero-flux boundary condition called also Neumann boundary condition

$$(f(u) - \nabla\phi(u)).\eta(x) = 0 \text{ on } F. \tag{3}$$

b) Robin boundary condition

$$b(u) - (f(u) - \nabla\phi(u)).\eta(x) = 0 \text{ on } F. \tag{4}$$

c) Dirichlet boundary condition

$$u(t,x) = u^D \text{ on } F. \tag{5}$$

Equation (1) is said to be degenerate because of the function ϕ which can cancel according to values of the unknown u. It is assumed that the solution of interest is confined within an interval $[0; u_{\max}]$ where $u_{\max}$ is a maximum value of the solution. We assume that there is also a critical value of u, denoted u_c such that for values in $[0, u_c]$ the problem becomes hyperbolic, i.e. $\phi \equiv 0$ and beyond this critical value, the function ϕ is strictly increasing. There is a vast literature for degenerate parabolic problems, [1], [2], [3], [4], [5], [6], [7], [8]. These equations are found in several types of the field of phenomena modelling the mechanics of continuous environments, problems of population dynamics, in petroleum engineering, the sedimentation process of clay suspensions in closed containers and, in population dynamics, the stochastic migration of a group of animals (prey or predator) in a specific limited area, [9], [10], [11], [12], [13], [14], [15]. In (1), we suppose that the convection flux function f is Lipschitz-continuous. Furthermore, we assume, according to the zero-flux boundary condition, (3) and Dirichlet boundary condition (5), that

$$f(0) = 0,\ f(u_{\max}) = 0 \text{ for some } u_{\max} > 0 \tag{6}$$

and we assume also that b in Robin boundary condition (4) satisfies the following hypotheses:

$$f(0) = 0,\ b(0) = 0\ \ b(u_{\max}) \geq |f(u_{\max}).\eta| \tag{7}$$

$$b = \beta \circ \phi \tag{8}$$

where β in (8) is a Lipschitz continuous function and non-decreasing. Notice that in (2), we take

$u_0 \in [0, u_{\max}]$. We work in a purely numerical setting by presenting numerical illustrations. The numerical tests presented in this article are obtained using an explicit in time finite volume scheme under the appropriate CFL (Courant-Friedrichs-Lewy) conditions (The reader can consult the following references to fully understand the finite volumes method, [16], [17], [18], [19], [20]). We also study the convergence rate of the scheme. Our goal is to understand the role of certain assumptions ((6), (7), (8)) considered in the theoretical study by the first author of this paper, [21], [22], [23], and to verify the robustness of our scheme. Our scheme in 1D is

$$\begin{aligned} u_i^{n+1} =& u_i^n - \frac{\delta t}{\delta x}\left(F\left(u_i^n, u_{i+1}^n\right) - F\left(u_{i-1}^n, u_i^n\right)\right) \\ &+\frac{\delta t}{\delta x^2}\left(\phi\left(u_{i+1}^n\right) + b\left(u_{i-1}^n\right) - 2b\left(u_i^n\right)\right) \quad (9) \end{aligned}$$

In (9), we consider a finite difference scheme to approximate the diffusion flux ϕ. For the approximation of convection flux, we used Rusanov flux as well as Lax-Friedrichs numerical flux.
This allows us to compare theory and numerics and to consider other assumptions in future work. The observations below concern both the pure hyperbolic (PH) and the degenerate parabolic (DP) case. Our paper is subdivided into three parts. In section 2, we discuss the evolution and behavior of numerical solutions and the convergence rate of the scheme. In the last part, we are interested in the role of containment hypotheses in the evolution of solutions.

2 Evolution and behavior of numerical solutions and convergence rate of the scheme

2.1 Illustrations and numerical behaviors of the solution's evolution

Here we present numerical tests to observe the evolution of the numerical solution of the equation $u_t + f(u)_x - \phi(u)_{xx} = 0$ in hyperbolic regimes ($\phi \equiv 0$) and in degenerate parabolic regimes (ϕ continuous, non-decreasing function). We examine the three boundary conditions (zero-flux, Robin and Dirichlet). As mentioned above, the tests are carried out using the CFL condition, but note that the CFL condition is more restrictive in the degenerate parabolic case. We take $\delta x = 0,01$ and $\delta t = 210^{-3}$ (for degenerate parabolic case) and $\delta t = 210^{-5}$ (for hyperbolic case).

2.1.1 Zero-flux boundary condition

Here, we consider an interval $I = (0, 1)$. The functions f and ϕ given by Table 1. Let us consider the following initial condition:

$$u_0(x) = \begin{cases} 0.4 & \text{if } x \leq 0.5 \\ 0.9 & \text{if } x \geq 0.5 \end{cases} \quad (10)$$

We can observe that a portion (at the bottom left, see Figure 1) of the solutions for the degenerate parabolic and hyperbolic cases coincides. Note also that the hyperbolic solution is constrained to reach the value 1 at the point $x = 1$. In contrast, the parabolic solution additionally uses the contribution from the slope $\phi(u)_x$ of $\phi(u)$ to satisfy the boundary condition. We also observe that the shock in the parabolic region disappears instantaneously, while the shock in the hyperbolic area is maintained, and the impact speed is very different; even the slope of $\phi(u)$ on the right side of the shock contributes significantly to the speed (it is clear that the shock propagates much faster here in the degenerate parabolic case compared to the hyperbolic case).

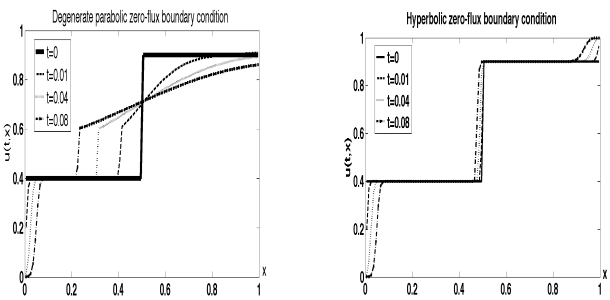

Fig. 1: Evolution and behavior of the numerical solution with a zero-flux boundary condition

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Table 1: "The data used to illustrate Figure 1"

Problem type.	f	ϕ
DP	$v(1-v)$	$(v-0.6)^+$
PH	$v(1-v)$	0

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2.1.2 Robin boundary condition

With a Robin boundary condition, we observe with the same previous initial condition (10) a similar illustration of the evolution of the solution as in the zero-flux degenerate parabolic case (see the left part of Figure 2). Indeed, in this case $b(u) = \phi(u) = (u - u_c)^+$ (Table 2) and in the vicinity of $x = 0$, due to the choice of the value of u_c (taking $u_c = 0.6$), we can see very clearly that $b(u)$ is zero and that the boundary problem then coincides with the zero-flux condition. Naturally, in the vicinity of $x = 1$ where u takes values above the degeneracy threshold $u_c = 0.6$, we observe a behavior typically analogous to the case of zero-flux boundary condition but the numerical solution is quantitatively different. For the purely hyperbolic case, the hypothesis (8) forces $b(u)$ to be zero, so Robin's case coincides with the zero-flux case. Let us consider, as a second example, the framework of [24], "by taking $b(u) = f(u).\eta$, with $\eta(x) = -1$ to $x = 0$ and $\eta(x) = +1$ to $x = 1$". According to their theoretical results, if $f.\eta$ is monotone and non-decreasing (this requires a $x-$dependence also in the $f = f(x, u)$ stream), there is a unique solution to the corresponding Robin problem that corresponds, in fact, to the Neumann boundary condition $\partial_x \phi(u).\eta = 0$. The solutions satisfy the maximum principle with respect to the maximum and minimum values of u_0. In our case, we have taken x-independent $f(u)$, which means that the assumptions of [24], are not satisfied. Nevertheless, the scheme works well, and we observe that the numerical solution converges to a limit that satisfies the maximum principle (see the right part of Figure 2).

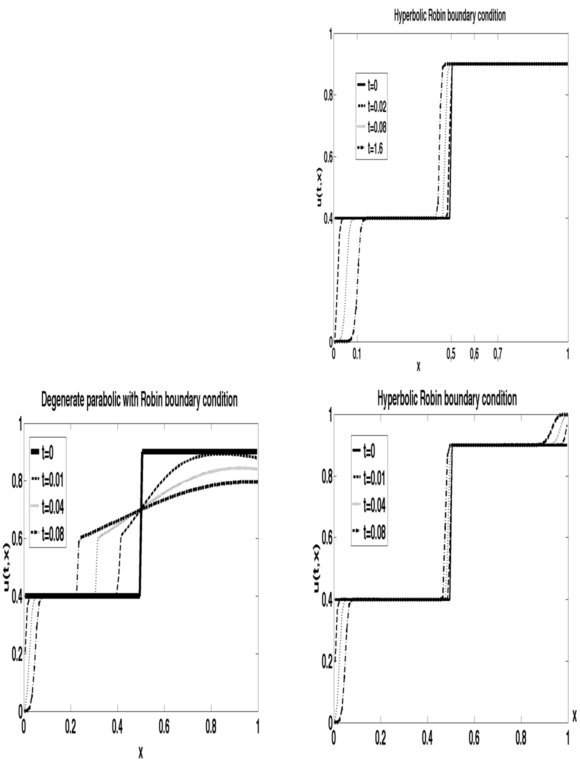

Fig. 2: Evolution and behavior of the solution with Robin-type boundary condition in $1D$ with $b(u) = \phi(u)$

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Table 2: "The data used to illustrate Figure 2"

Problem.	f	ϕ	b
DP	$v(1-v)$	$(v-0.6)^+$	$(v-0.6)^+$
PH	"$v(1-v)$"	0	"$f(v).\eta$"

Source:Created by the authors

From now, if we change the function b (Table 3), a clear difference is observed in Figure 3).

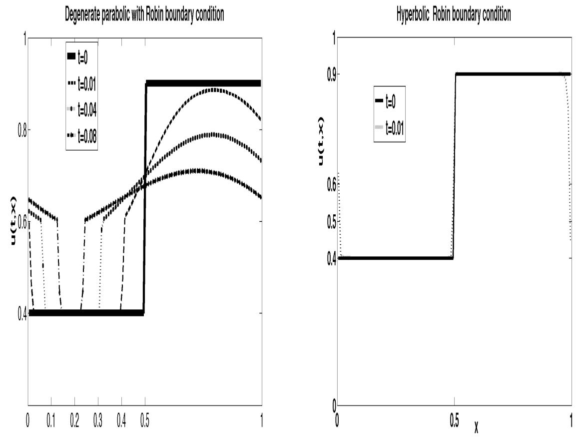

Fig. 3: Evolution of solution with Robin boundary condition in $1D$, with $b(u) = u$

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Table 3: "The data used to illustrate Figure 3"

Problem.	f	ϕ	b
DP	$v(1-v)$	$(v-0.6)^+$	$v.\eta$
PH	$v(1-v)$	$\phi(v) = 0$	$v.\eta$

Source:Created by the authors

2.1.3 Dirichlet boundary condition

We deal, in the case of a Dirichlet boundary condition, with the evolution of initial data. In PDEs framework, there are many articles addressing this type of boundary condition, ([25], [26], [27], [28]. Our goal is to illustrate certain phenomena that the first author of this paper described in [23], "for example with f and ϕ given by Table 4 the fact that in Figure 4, we observe at $x = 1$ the formation of a boundary layer, and the solution cannot satisfy the formal boundary condition. The solution remains bounded between the maximum and minimum values of the initial and boundary data without any containment assumption", as explained in [23].

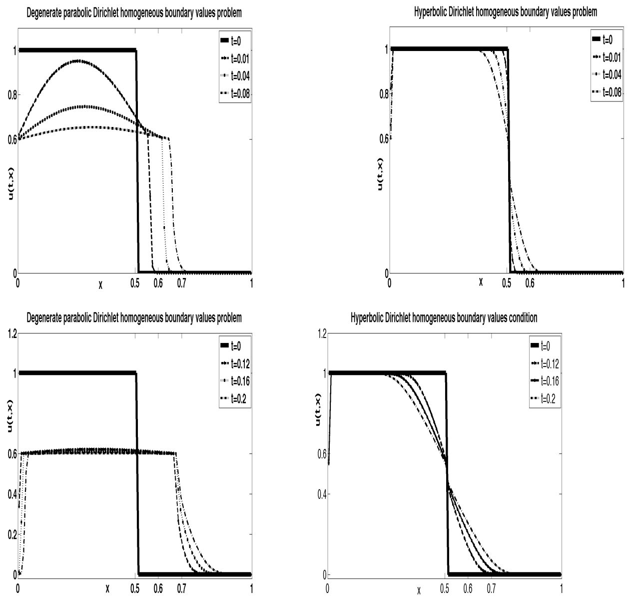

Fig. 4:Evolution and behavior of the solution with homogeneous Dirichlet boundary conditions

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Table 4: "The data used to illustrate the figure 4"

Problem.	f	ϕ
DP	$v(1-v)$	$(v-0.6)^{+}$
PH	$v(1-v)$	0

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2.2 Mesh refinement and convergence speed

We will deal only with the case of the zero-flux boundary condition (The same remarks apply to the other boundary conditions). The calculations for this test are based on the explicit finite volume scheme from [22]. The simulations stop at $t = 2$ s. The measurements of the numerical error provided as δx tends to zero are presented. To measure the error, we use the L^1 norm of the numerical error given by $\sum_{i=1}^{N} |u_i^{app} - u_i^{ref}|$. Several spatial grids are considered, involving cells with spatial steps δx equal to $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{40}$, $\frac{1}{80}$ and 10, 20, 40, 80, respectively. The reference function u_{ref} is the numerical solution with cells $\delta x = \frac{1}{160}$ and 160. The time step is chosen according to the CFL condition ($\delta t \leq C\delta x$ if ϕ is set to zero, and $\delta t \leq C\delta x^2$ otherwise). Figure 5 shows the logarithmic scale profile of the error.

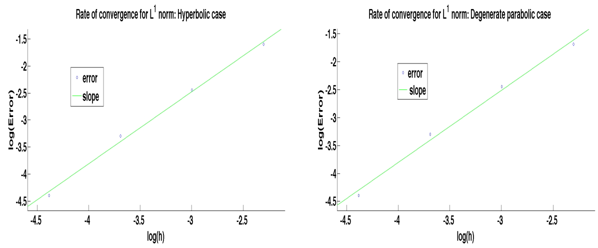

Fig. 5: Rate of convergence

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3 Numerical behavior of numerical solutions with and without containmemt assumptions

3.1 Other additional numerical illustrations

The importance of hypothesis (6) for taking into account the zero-flux boundary condition studied in [21], will be the focus of this section. We also offer an interpretation of the numerical observation for the study of the Robin boundary condition by analyzing in a meticulous and rigorous manner the role of the assumptions (8) and (7), mentioned in reference, [21].

"For the zero-flux boundary condition, the entropy formulation is based on the assumption (6)", [21]. This assumption allows the solution to be confined in the interval $[0, u_{\max}]$.

More specifically, if we assume that the initial data takes values in $[0, u_{\max}]$, then the interval $[0, u_{\max}]$ will be the invariant domain of the solution. Here, we assume $u_{\max} = 1$ and consider the initial data in (2) taking values in $[0, u_{\max}]$. Then, if (6) is satisfied, we observe that the numerical solution remains within this interval (see the first figure on the left of Figure 6 and Figure 7, for both the hyperbolic and degenerate parabolic cases). But if we remove the hypothesis (6) by considering for example in Table 5 with $f(u) = \frac{u^2}{2}$ (Burgers equation), then at a certain distance from the boundary the sequence of numerical solutions remains bounded in the degenerate case (the second to the right of Figure 7) and also for the hyperbolic case, however we also observe that there appears a kind of boundary layer at $x = 1$ (Figure 6). This numerical test shows that the containment hypothesis (6) might not be necessary to study the problem (1)-(3). We observed convergence to a bounded solution, but not in the sense of our definition of an entropy solution. In future work, we hope to find an appropriate formulation leading to

existence and uniqueness results without (6).
Second, in our Robin boundary condition framework, considering $f(0) = b(0) = 0$, the hypothesis (7) plays the same role as the containment hypothesis in the sense that it allows us to have the numerical solution in $[0, u_{\max}]$. We notice that in the situation where (7) is deleted and $u_0 \in [0, u_{\max}]$ then the numerical solution does not satisfy the entropy formulation, because $u_h \notin [0, u_{\max}]$. In addition, our entropy formulation requires us to choose $b(u)$ in the functional space that allows us to define the trace of $b(u)$ on the boundary; for this reason, the hypothesis (8) was imposed in [22].
In the context where the hypothesis (8) is not taken into account as in Table 6, numerically, a boundary layer is observed (see Figure 8). Now, taking these assumptions into account, numerical observation seems to illustrate well the theoretical results on the well-possedness of the problem (1)-(4) (see Figure 8). We expect that with this assumption, the boundary condition will be exactly satisfied, and if the numerical solution converges to the unique entropy solution, then we avoid a boundary layer.

Table 5: "The data used to illustrate Figures 6 and Figure 7 for zero-flux problems"

Case.	f	ϕ
DP without (6)	$\frac{v^2}{2}$	$(v-0.6)^+$
PH without (6)	$\frac{v^2}{2}$	0
DP with (6)	$v(1-v)$	$(v-0.6)^+$
PH with (6)	$v(1-v)$	0

Source:Created by the authors

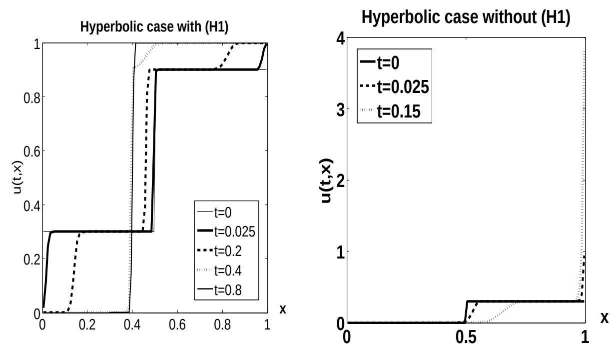

Fig. 6: "Hyperbolic zero-flux problem: with and without hypothesis (6) denoted by (H1)"

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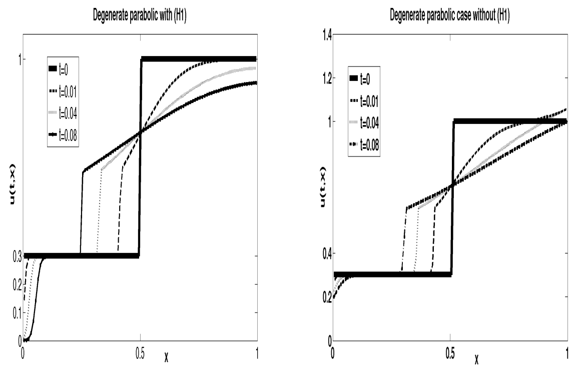

Fig. 7: "Degenerate parabolic zero-flux problem: with and without hypothesis (6) denoted by (H1)"

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Table 6: "The data used to illustrate Figure 8 for degenerate parabolic equation ($\phi(v) = (v - 0.6)^+$) with Robin boundary condition"

Case.	f	$b(v)$
without (8)	$\frac{v^2}{2}$	v
without (7)	$\frac{v^2}{2}$	$(v-0.6)^+$
with (7) and (8)	$v(1-v)$	$(v-0.6)^+$
with (7) and (8)	$v(1-v)$	v

Source:Created by the authors

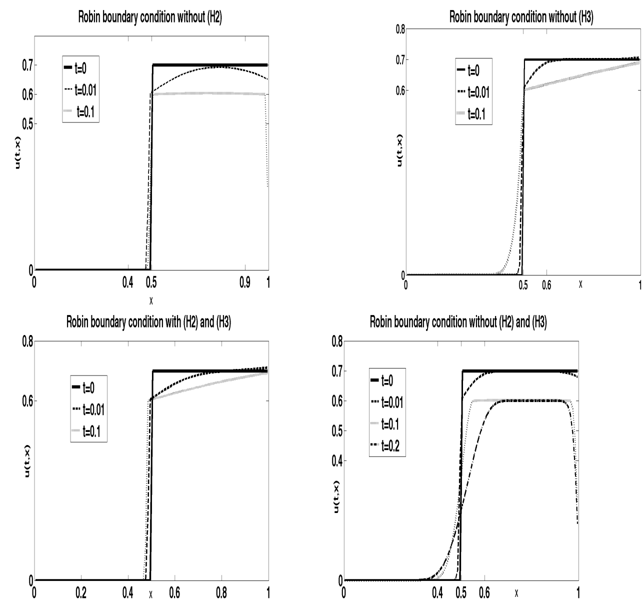

Fig. 8: Assumption (8) denoted by (H3) and assumption (7) denoted by (H2) in the case of Robin boundary problem

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3.2 Conjectures and perspectives

First, let us describe several theoretical perspectives.

• These observations seem to indicate that in the absence of hypotheses (6) (resp. (7)), there exists a bounded entropy solution for zero-flux at the boundaries (and the Robin problem), one can refer to [29], for the special hyperbolic problem, where an entropy solution formulation was proposed, with an indirect justification of the existence question. If we consider the sequence of approximate solutions, we realize that we cannot obtain $L^\infty-$estimates if we do not take into account hypotheses (6) or (7). This results in the difficulty of achieving convergence of the sequence of approximate solutions. It is then necessary to look for a new entropy formulation where an obstacle-type condition will be established at the point u_c.

Bounded entropy solution to the zero-flux problem seems to exist without (6) (without (7), for the Robin problem). We refer, in particular, to the hyperbolic case studied in [29], where an entropy formulation was proposed, with an indirect justification of the existence question. Due to the absence of a global $L^\infty-$ estimate on the approximate solutions (because of the lack of (6) or (7)) convergence of numerical approximations has not yet been proved, even for the pure hyperbolic case of [29]. In general, it remains to find an appropriate entropy formulation for the degenerate parabolic problem, possibly using ideas from the new formulation of the Dirichlet case, [23], where the role of the obstacle-type condition at the point u_c was highlighted.

• Assumption (8) in the Robin case ensures that the value $b(u)$ present in the integral on the boundary in entropy formmulaion of [22], makes sense, defined as $b(u)|_{(0,T)\times\partial I} = \beta(b(u)|_{(0,T)\times\partial I})$. In this case, according to Lemma 1.4 of [22], we observe the absence of a boundary layer in the numerical sequence of solutions. By relaxing (8), We can always define the trace of $b(u)$ as in the paper, [30]. But as shown by the numerical experiments above, a boundary layer can develop without the hypothesis (8). Moreover, (8) was used in the proof of convergence of the approximate solutions in the article, [21]. "Thus, although we believe that the entropy formulation of [21], for the Robin boundary condition is appropriate even in the absence of hypothesis (8), the existence theorem for the Robin case without (8) remains to be proven".

In conclusion, let us briefly present some numerical perspectives that deserve to be studied. It would be interesting, to avoid CFL conflicts, to propose implicit schemes over time.

• In nature, there are mathematical models that take into account mixed conditions (for example, on the domain $[0, 1]$ at the point $x = 1$ a zero-flux condition and at the point $x = 0$ a Dirichlet condition). One could be interested in numerical observation with or without a containment hypothesis. Such a study could also inspire the theoretical study of this problem.

• A physical reality is the case of dynamic boundary conditions. This assumes that the boundary condition is described by an evolution equation over time. As a prelude to a theoretical study, a numerical discretization can allow for detailed observations of the evolution of the numerical solution and assess possibility or necessity of defining containment hypotheses that guarantee certain properties of solution.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The first author is the initiator of this project. The other two authors contributed significantly and equally to the improvement of the numerical simulations as well as to their interpretation. It can be stated that all three authors contributed equally to the final version of this article.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.