Analysis of Linear Canonical Ambiguity Functions in Besov-type Spaces

AYŞE SANDIKÇI
Department of Mathematics
Ondokuz Mayıs University
Faculty of Science, 55200, Atakum, Samsun
TURKEY

Abstract: - The classical ambiguity function, an essential time-frequency analytic tool, was developed to examine the time-frequency properties of non-stationary signals. Key radar waveform characteristics, such as time-delay resolution, Doppler resolution, and Doppler-tolerance, can be accurately and efficiently analyzed using this function. This paper investigates the continuity properties of the linear canonical ambiguity function, which is a generalization of the classical ambiguity function based on the linear canonical transform. It is demonstrated that the linear canonical ambiguity function yields significant results regarding continuity within Besov spaces and weighted Besov spaces. Furthermore, this work explores the Besov and weighted Besov distances between two linear canonical ambiguity functions associated with different functionals and signals.

Key-Words: - Ambiguity function, linear canonical transform, linear canonical ambiguity function, Besov Space, weighted Besov space

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1 Introduction

One of the most widely utilized techniques in signal processing and many other scientific domains is Fourier analysis. The fractional Fourier transform (FrFT), a generalization of the classical Fourier transform (FT), was introduced to the mathematical literature in the 1980s. First introduced by Wiener in 1929, [1], the FrFT was designed to solve the quantum mechanical differential equation. Optical problems can also be interpreted with the FrFT. The FrFT has been used in many scientific disciplines such as signal and image processing, communications, wave propagation. The subject came into the limelight with the work of Namias in 1980, [2]. Many ideas related to the FT have been extended to the linear canonical transform (LCT) domain as a further expansion of the FRFT, which is demonstrated to have a significant impact on optics and signal processing. First-order optical systems, which are theoretically similar to the linear canonical transformations (LCTs), include transmission through narrow lenses, propagation across quadratic graded-index media, free space propagation under the Fresnel approximation, and their arbitrary combinations. LCT is more adaptable due to the additional degrees of freedom, and it has been demonstrated to be an effective tool for filter construction, time-frequency analysis, phase retrieval, pattern recognition, etc.

The ambiguity function (AF) is a significant instrument employed in the field of signal processing for the purpose of delineating methodologies that facilitate the recovery of amplitude and phase information from intensity measurements. In accordance with the traditional concept of ambiguity function (AF), Pei and Ding [3] first examine the linear canonical ambiguity function (LCAF) and derive several significant characteristics. In the field of optical signal processing, Zhao et al. [4] have recently looked at the characteristics and physical significance of the LCAF. By the paper Che, Li and Xu, [5], was presented a novel definition for LCAF which differs to the definitions of AF provided in [3], [4]. The key characteristics and use of the novel AF in the linear frequency-modulated signal processing were also covered.

This study analyzes the continuity of the LCAF, as introduced in [5], within the framework of both Besov and weighted Besov settings, focusing on their respective norm topologies and the influence of temperate weight functions.

PROOF
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1.1 Preliminary

The Linear Canonical Transform (LCT): Let

$$\Theta=(a,b;c,d)=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in R^{2\times 2}$$

be a matrix parameter satisfying $det\Theta = ad-bc = 1$. The LCT of a signal $h\in L^1(R)$ is defined by

$$L_\Theta\big(h(t)\big)(u)=\begin{cases}\int_R h(t)\ K_\Theta(t,u)dt, & b\neq 0\\ \sqrt{d}e^{i\frac{c}{d}u^2}h(du), & b=0\end{cases}$$

where $K_\Theta(t,u)$ is so-called the kernel of the LCT given by

$$K_\Theta(t,u)=\frac{1}{\sqrt{2\pi bi}}\ e^{\frac{i}{2b}(at^2-2tu+du^2)}.$$

It is clear from the LCT definition that a signal's LCT is basically a chirp multiplication when $b=0$. Consequently, in this study, the assumption is made that $b\neq 0$, [3], [4], [5].

The cross ambiguity function (AF) of functionals h and f, which are elements of the $L^2(R)$ space, is defined as follows

$$A(h,f)(u,\mu)=\int_R e^{-i\mu v}\, h\left(v+\frac{u}{2}\right)\overline{f\left(v-\frac{u}{2}\right)}\ dv,$$

if h is equal to f, then $Ah(u,\mu)$ is referred to the autoambiguity function of h is

$$A(h,f)(u,\mu)=\int_R e^{-i\mu v}\, h\left(v+\frac{u}{2}\right)\overline{h\left(v-\frac{u}{2}\right)}\ dv.$$

It is important to note that the term "ambiguity functions" is used to refer to both autoambiguity and cross ambiguity functions, [6].

By substituting the kernel of the FT with the kernel of the LCT in the AF definition, we may derive a definition of the linear canonical ambiguity function (LCAF) from the definition of the AF associated with the Fourier transform (FT), [3], [4], [5]:

$$A^\Theta(h,f)(u,\mu) = \int_R K_\Theta(v,\mu)h\left(v+\frac{u}{2}\right)\overline{f\left(v-\frac{u}{2}\right)}\ dv.$$

Let k be a positive function on R. If there exist $C>0$ and $N>0$ such that

$$k(u+\mu)\le (1+C|u|)^N k(\mu),\quad u,\mu\in R,$$

it is referred to as a ‘temperate weight function’. In this context, the symbol Ω will represent the collection of all such functions.

For $1\le p<\infty$, the space $L_k^p(R)$ consists of all complex-valued measurable functions on R that satisfy

$$\int_R |h(y)|^p k(y)dy<\infty.$$

We define $L_k^p(R)$ norm of h by

$$\|h\|_{L_k^p}=\|h\|_{p,k}=\left(\int_R |h(y)|^p k(y)dy<\infty\right)^{\frac{1}{p}}$$

Then $L_k^p(R)$ is complete with the norm $\|\cdot\|_{p,k}$. If $k=1$, then the space $L_k^p(R)$ is the Lebesgue space $L^p(R)$.

Besov spaces $B_p^{\beta,q}(R)$ play a pivotal role in various fields of mathematical analysis and applied mathematics, primarily due to their fine-tuning capability in measuring the regularity of functions. Unlike standard Sobolev spaces, the additional parameter q allows for a more precise characterization of functions with localized irregularities.

Besov spaces serve as a bridge between other function spaces. They are frequently used in the study of Fourier multipliers and the boundedness of operators. Because they can be defined via Littlewood-Paley theory (frequency decomposition), they allow analysts to study functions by looking at their behavior across different frequency scales.

Let $p\in[1,\infty]$ and $\beta\in(0,1)$. For $q\in[1,\infty[$ the Besov space $B_p^{\beta,q}(R)$ is defined as the subspace of functions $h\in L^p(R)$ such that the following semi-norm is finite:

$$[h]_{B_p^{\beta,q}}:=\left(\int_R \frac{\|h(\cdot+s)\ -h(\cdot)\|_p^q}{|s|^{1+\beta\ q}}\ ds\right)^{1/q}<\infty.$$

In the limiting case where $q=\infty$, the Besov space $B_p^{\beta,\infty}(R)$ consists of all functions $h\in L^p(R)$ for which the semi-norm is defined via the essential supremum:

$$[h]_{B_p^{\beta,\infty}}:=\sup_{s\neq 0}\frac{\|h(\cdot+s)\ -h(\cdot)\|_p}{|s|^\beta}<\infty.$$

The full norm for these spaces is given by

$$\|h\|_{B_p^{\beta,q}}=\|h\|_p+[h]_{B_p^{\beta,q}}.$$

Here, the L^p-modulus of continuity is denoted by

$$w_p(h,s)=\|h(\cdot+s)\ -h(\cdot)\|_p$$

which is well-defined and bounded by $w_p(h,s)\le 2\|h\|_p$ due to the triangle inequality.

The weighted Besov space $B^{\beta,q}_{p,k}(R)$ can be characterized as a generalization of the classical Besov space, where the standard Lebesgue measure in the L^p norm is replaced by the weighted measure associated with a temperate weight function k. Specifically, a function h belongs to $B^{\beta,q}_{p,k}$ if the L^p_k norm of its finite differences satisfies the Besov-type integrability condition, [7], [8], [9].

2 Main Results

In this section, we investigate the continuity properties of the linear canonical ambiguity function, a generalization of the classical ambiguity function, within the framework of Besov spaces $B^{\beta,q}_{p}(R)$ and weighted Besov spaces $B^{\beta,q}_{p,k}(R)$. Specifically, we focus on the case where $p = q = 1$ over the domain $[c,\infty) \subset R$, $c > 0$.

Theorem 1. Let $f \in L^1([c,\infty[)$. The function $h \to A^{\Theta}(h,f)$ defines a continuous operator from $B^{\beta,1}_1([c,\infty[)$ to itself. Moreover

$$\|A^{\Theta}(h,f)(\cdot,\mu)\|_{B^{\beta,1}_1} \leq \frac{1}{|2\pi b|}\left(\|h\|_{B^{\beta,1}_1} + (K+1)\|h\|_1\right)\|f\|_1,$$

where $K = 2/\beta c^{\beta}$.

Proof. If $h \in B^{\beta,1}_1([c,\infty[)$, then $h \in L^1([c,\infty[)$. We assert that $A^{\Theta}(h,f)(\cdot,\mu)$ belongs to the space $B^{\beta,1}_1([c,\infty[)$. It is known from [10] that the LCAF can also be represented in the following form:

$$A^{\Theta}(h,f)(u,\mu) = \frac{1}{2\pi b}\, e^{\frac{i}{2b}u(au-\mu)} \int_R h(m+u)\overline{f(m)}\, e^{\frac{i}{b}m(au-\mu)} dm$$

By using this equality and the Young inequality, we write

$$\|A^{\Theta}(h,f)(\cdot,\mu)\|_1 \leq \frac{1}{|2\pi b|}\int_R \left|\overline{f(m)}\right| \left(\int_R |h(m+u)|\, du\right) dm = \frac{1}{|2\pi b|}\|f\|_1\|h\|_1. \tag{1}$$

On the other hand

$$\begin{aligned}
&w_1\left(A^{\Theta}(h,f)(\cdot,\mu),s\right) = \left\|A^{\Theta}(h,f)(\cdot+s,\mu) - A^{\Theta}(h,f)(\cdot,\mu)\right\|_1 \\
&= \int_{[c,\infty[} \left|A^{\Theta}(h,f)(u+s,\mu) - A^{\Theta}(h,f)(u,\mu)\right| du \\
&= \frac{1}{|2\pi b|}\int_{[c,\infty[} \left| \int_{[c,\infty[} h(m+u+s)\overline{f(m)}\, e^{\frac{i}{b}m(a(u+s)-\mu)} e^{\frac{i}{2b}(u+s)(a(u+s)-\mu)} dm - \int_{[c,\infty[} h(m+u)\overline{f(m)}\, e^{\frac{i}{b}m(au-\mu)} e^{\frac{i}{2b}u(au-\mu)} dm \right| du \\
&= \frac{1}{|2\pi b|}\int_{[c,\infty[} \left| \int_{[c,\infty[} \overline{f(m)}\, e^{\frac{i}{b}m(au-\mu)} e^{\frac{i}{2b}u(au-\mu)} \left(h(m+u+s) e^{\frac{i}{b}mas} e^{\frac{i}{2b}(2uas-\mu s+as^2)} - h(m+u)\right) dm \right| du \\
&\leq \frac{1}{|2\pi b|}\int_{[c,\infty[}\int_{[c,\infty[} |f(m)| \left| \left(h(m+u+s) - h(m+u)\right) + \left(e^{\frac{i}{b}mas} e^{\frac{i}{2b}(2uas-\mu s+as^2)} - 1\right) h(m+u) \right| dm\, du \\
&\leq \frac{1}{|2\pi b|}\int_{[c,\infty[} |f(m)| \left(\int_{[c,\infty[} |h(m+u+s) - h(m+u)| du + \int_{[c,\infty[} \left(e^{\frac{i}{b}mas} e^{\frac{i}{2b}(2uas-\mu s+as^2)} - 1\right) h(m+u)\, du \right) dm \\
&\leq \frac{1}{|2\pi b|}\int_{[c,\infty[} |f(m)| (w_1(h,s) + 2\|h\|_1)\, dm \\
&= \frac{1}{|2\pi b|}\|f\|_1 (w_1(h,s) + 2\|h\|_1).
\end{aligned}$$

Hence, we get for $\beta \in (0,1)$

$$\begin{aligned}
&\left[A^{\Theta}(h,f)(\cdot,\mu)\right]_{B^{\beta,1}_1} = \int_{[c,\infty[} w_1\left(A^{\Theta}(h,f)(\cdot,\mu),s\right)\frac{ds}{s^{1+\beta}} \\
&\leq \frac{1}{|2\pi b|}\|f\|_1 \int_{[c,\infty[} (w_1(h,s) + 2\|h\|_1)\frac{ds}{s^{1+\beta}} \\
&= \frac{1}{|2\pi b|}\|f\|_1 \left(\int_{[c,\infty[} w_1(h,s)\frac{ds}{s^{1+\beta}} + 2\|h\|_1 \int_{[c,\infty[} \frac{ds}{s^{1+\beta}} \right) \\
&= \frac{1}{|2\pi b|}\|f\|_1 \left(\int_{[c,\infty[} w_1(h,s)\frac{ds}{s^{1+\beta}} + K\|h\|_1 \right),
\end{aligned}$$

where $K = 2/\beta c^{\beta}$. It follows from the last integral and (1)

$$\|A^{\Theta}(h,f)(\cdot,\mu)\|_{B^{\beta,1}_1} = \|A^{\Theta}(h,f)(\cdot,\mu)\|_1 + \left[A^{\Theta}(h,f)(\cdot,\mu)\right]_{B^{\beta,1}_1}$$

$$= \left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_1 + \int_{[c,\infty[} w_1\left(A^{\Theta}(h,f)(\cdot,\mu),s\right)\frac{ds}{s^{1+\beta}}$$
$$\le \frac{1}{|2\pi b|}\|f\|_1\|h\|_1 + \frac{1}{|2\pi b|}\|f\|_1\left(\int_{[c,\infty[} w_1(h,s)\frac{ds}{s^{1+\beta}} + K\|h\|_1\right)$$
$$\le \frac{1}{|2\pi b|}\|f\|_1\left(\|h\|_1 + \|h\|_{B_1^{\beta,1}} + K\|h\|_1\right)$$
$$= \frac{1}{|2\pi b|}\|f\|_1\left(\|h\|_{B_1^{\beta,1}} + (K+1)\|h\|_1\right).$$

Theorem 2. Let $\chi,\rho \in L^1([c,\infty[)$. If $h_1,\ h_2 \in B_1^{\beta,1}([c,\infty[)$, then
$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu) - A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}}$$
$$\le \frac{1}{|2\pi b|}\|\chi-\rho\|_1\left(\|h_1\|_{B_1^{\beta,1}} + (K+1)\|h_1\|_1\right) + \frac{1}{|2\pi b|}\|\rho\|_1\left(\|h_1-h_2\|_{B_1^{\beta,1}} + (K+1)\|h_1-h_2\|_1\right),$$
where $K = 2/\beta c^{\beta}$.

Proof. We write
$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu) - A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}}$$
$$\le \left\|A^{\Theta}(h_1,\chi)(\cdot,\mu) - A^{\Theta}(h_1,\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}} + \left\|A^{\Theta}(h_1,\rho)(\cdot,\mu) - A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}}.$$
It is evident that
$$A^{\Theta}(h_1,\chi)(u,\mu) - A^{\Theta}(h_1,\rho)(u,\mu)$$
$$= \frac{1}{2\pi b}\, e^{\frac{i}{2b}u(au-\mu)}\int_R h_1(m+u)\,\overline{(\chi-\rho)(m)}\, e^{\frac{i}{b}m(au-\mu)}dm$$
$$= A^{\Theta}(h_1,\chi-\rho)(u,\mu)$$
and similarly
$$A^{\Theta}(h_1,\rho)(u,\mu) - A^{\Theta}(h_2,\rho)(u,\mu) = A^{\Theta}(h_1-h_2,\rho)(u,\mu).$$
Then by Theorem 1, we write
$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu) - A^{\Theta}(h_1,\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}}$$
$$= \left\|A^{\Theta}(h_1,\chi-\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}}$$
$$\le \frac{1}{|2\pi b|}\|\chi-\rho\|_1\left(\|h_1\|_{B_1^{\beta,1}} + (K+1)\|h_1\|_1\right)$$
and
$$\left\|A^{\Theta}(h_1,\rho)(\cdot,\mu) - A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}}$$
$$= \left\|A^{\Theta}(h_1-h_2,\rho)(\cdot,\mu)\right\|_{B_1^{\beta,1}}$$
$$\le \frac{1}{|2\pi b|}\|\rho\|_1\left(\|h_1-h_2\|_{B_1^{\beta,1}} + (K+1)\|h_1-h_2\|_1\right).$$
This completes the proof.

Theorem 3. Let $f \in L^1([c,\infty[)$ be a fix function such that
$$\int_{[c,\infty[} |f(m)|\,(1+C|m|)^N dm \le M < \infty.$$
Then $A^{\Theta}(\cdot,f): B_{1,k}^{\beta,1}([c,\infty[) \to B_{1,k}^{\beta,1}([c,\infty[),\quad h \to A^{\Theta}(h,f)$ is continuous and it holds
$$\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{B_{1,k}^{\beta,1}} \le \frac{M}{|2\pi b|}\left(\|h\|_{B_{1,k}^{\beta,1}} + (K+1)\|h\|_{1,k}\right),$$
where $K = 2/\beta c^{\beta}$.

Proof. If $h \in B_{1,k}^{\beta,1}([c,\infty[)$, then $h \in L_k^1([c,\infty[)$. It is claimed that $A^{\Theta}(h,f)(\cdot,\mu)$ is an element of $B_{1,k}^{\beta,1}([c,\infty[)$. By using the equality
$$A^{\Theta}(h,f)(u,\mu) = \frac{1}{2\pi b}\, e^{\frac{i}{2b}u(au-\mu)}\int_R h(m+u)\overline{f(m)}\, e^{\frac{i}{b}m(au-\mu)}dm$$
and the Young inequality, we write
$$\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{1,k} = \int_{[c,\infty[}\left|A^{\Theta}(h,f)(u,\mu)\right|k(u)du$$
$$\le \frac{1}{|2\pi b|}\int_R\int_R\left|\overline{f(m)}\right|\left(\int_R |h(m+u)|k(u)\,du\right)dm.$$
By changing variable $m+u=z$ and using the inequality $k(z-m) \le (1+C|m|)^N k(z)$, we have
$$\left\|A^{\Theta}(h,f)(\cdot,\mu)\right\|_{1,k} \tag{4}$$
$$\le \frac{1}{|2\pi b|}\int_R\left|\overline{f(m)}\right|\left(\int_R |h(z)|k(z-m)\,dz\right)dm$$
$$\le \frac{1}{|2\pi b|}\int_R |f(m)|(1+C|m|)^N\left(\int_R |h(z)|k(z)\,dz\right)dm$$
$$= \frac{1}{|2\pi b|}\|h\|_{1,k}\int_R |f(m)|(1+C|m|)^N\,dm$$

$$\leq \frac{M}{|2\pi b|}\|h\|_{1,k}.$$

On the other hand, we write

$$w_{1,k}\big(A^{\Theta}(h,f)(\cdot,\mu),s\big) = \big\|A^{\Theta}(h,f)(\cdot+s,\mu) - A^{\Theta}(h,f)(\cdot,\mu)\big\|_{1,k}$$
$$= \int_{[c,\infty[}\big|A^{\Theta}(h,f)(u+s,\mu) - A^{\Theta}(h,f)(u,\mu)\big|k(u)du$$
$$= \frac{1}{|2\pi b|}\int_{[c,\infty[}\Bigg|\int_{[c,\infty[} h(m+u+s)\overline{f(m)}\,e^{\frac{i}{b}m(a(u+s)-\mu)}e^{\frac{i}{2b}(u+s)(a(u+s)-\mu)}dm - \int_{[c,\infty[} h(m+u)\overline{f(m)}\,e^{\frac{i}{b}m(au-\mu)}e^{\frac{i}{2b}u(au-\mu)}dm\Bigg|k(u)\,du$$
$$= \frac{1}{|2\pi b|}\int_{[c,\infty[}\Bigg|\int_{[c,\infty[}\overline{f(m)}\,e^{\frac{i}{b}m(au-\mu)}e^{\frac{i}{2b}u(au-\mu)}\Big(h(m+u+s)e^{\frac{i}{b}mas}e^{\frac{i}{2b}(2uas-\mu s+as^2)} - h(m+u)\Big)dm\Bigg|k(u)du$$
$$\leq \frac{1}{|2\pi b|}\int_{[c,\infty[}\int_{[c,\infty[}|f(m)|\,\Big|\big(h(m+u+s) - h(m+u)\big) + \Big(e^{\frac{i}{b}mas}e^{\frac{i}{2b}(2uas-\mu s+as^2)} - 1\Big)h(m+u)\Big|\,k(u)dm\,du,$$

Using the inequality $k(u) = k(u+m-m) \leq (1+C|m|)^N k(m+u)$, we get

$$w_{1,k}\big(A^{\Theta}(h,f)(\cdot,\mu),s\big) \leq \frac{1}{|2\pi b|}\int_{[c,\infty[}|f(m)|\,(1+C|m|)^N\Bigg(\int_{[c,\infty[}|h(m+u+s)-h(m+u)|k(m+u)du + \int_{[c,\infty[}\Big|\Big(e^{\frac{i}{b}mas}e^{\frac{i}{2b}(2uas-\mu s+as^2)} - 1\Big)h(m+u)\Big|\,k(m+u)du\Bigg)dm$$
$$\leq \frac{1}{|2\pi b|}\int_{[c,\infty[}|f(m)|(1+C|m|)^N\big(w_{1,k}(h,s) + 2\|h\|_{1,k}\big)\,dm$$
$$= \frac{1}{|2\pi b|}\big(w_{1,k}(h,s) + 2\|h\|_{1,k}\big)\int_{[c,\infty[}|f(m)|(1+C|m|)^N\,dm$$
$$\leq \frac{M}{|2\pi b|}\big(w_{1,k}(h,s) + 2\|h\|_{1,k}\big).$$

Hence, we obtain for $\beta \in (0,1)$

$$\big[A^{\Theta}(h,f)(\cdot,\mu)\big]_{B^{\beta,1}_{1,k}} \int_{[c,\infty[} w_{1,k}\big(A^{\Theta}(h,f)(\cdot,\mu),s\big)\frac{ds}{s^{1+\beta}}$$
$$\leq \frac{M}{|2\pi b|}\int_{[c,\infty[}\big(w_{1,k}(h,s) + 2\|h\|_{1,k}\big)\frac{ds}{s^{1+\beta}}$$
$$= \frac{M}{|2\pi b|}\Bigg(\int_{[c,\infty[} w_{1,k}(h,s)\frac{ds}{s^{1+\beta}} + 2\|h\|_{1,k}\int_{[c,\infty[}\frac{ds}{s^{1+\beta}}\Bigg)$$
$$= \frac{M}{|2\pi b|}\Bigg(\int_{[c,\infty[} w_{1,k}(h,s)\frac{ds}{s^{1+\beta}} + K\|h\|_{1,k}\Bigg),$$

where $K = 2/\beta c^{\beta}$. It follows from the last integral and (4)

$$\big\|A^{\Theta}(h,f)(\cdot,\mu)\big\|_{B^{\beta,1}_{1,k}} = \big\|A^{\Theta}(h,f)(\cdot,\mu)\big\|_{1,k} + \big[A^{\Theta}(h,f)(\cdot,\mu)\big]_{B^{\beta,1}_{1,k}}$$
$$= \big\|A^{\Theta}(h,f)(\cdot,\mu)\big\|_{1,k} + \int_{[c,\infty[} w_{1,k}\big(A^{\Theta}(h,f)(\cdot,\mu),s\big)\frac{ds}{s^{1+\beta}}$$
$$\leq \frac{M}{|2\pi b|}\|h\|_{1,k} + \frac{M}{|2\pi b|}\Bigg(\int_{[c,\infty[} w_{1,k}(h,s)\frac{ds}{s^{1+\beta}} + K\|h\|_{1,k}\Bigg)$$
$$\leq \frac{M}{|2\pi b|}\Big(\|h\|_{1,k} + \|h\|_{B^{\beta,1}_{1,k}} + K\|h\|_{1,k}\Big)$$
$$= \frac{M}{|2\pi b|}\Big(\|h\|_{B^{\beta,1}_{1,k}} + (K+1)\|h\|_{1,k}\Big).$$

Theorem 4. Suppose χ and ρ are functions in $L^1([c,\infty[)$ such that

$$\int_{[c,\infty[}|\chi(m)|\,(1+C|m|)^N dm \leq M_1 < \infty$$

and

$$\int_{[c,\infty[}|\rho(m)|\,(1+C|m|)^N dm \leq M_2 < \infty.$$

If $h_1,\ h_2 \in B^{\beta,1}_{1,k}([c,\infty[)$, then

$$\big\|A^{\Theta}(h_1,\chi)(\cdot,\mu) - A^{\Theta}(h_2,\rho)(\cdot,\mu)\big\|_{B^{\beta,1}_{1}}$$

$$\leq \frac{M_1+M_2}{|2\pi b|_1}\left(\|h_1\|_{B^{\beta,1}_{1,k}}+(K+1)\|h_1\|_{1,k}\right)+\frac{M_2}{|2\pi b|}\left(\|h_1-h_2\|_{B^{\beta,1}_{1,k}}+(K+1)\|h_1-h_2\|_{1,k}\right).$$

Proof. We write

$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu)-A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{B^{\beta,1}_{1,k}}$$
$$\leq \left\|A^{\Theta}(h_1,\chi)(\cdot,\mu)-A^{\Theta}(h_1,\rho)(\cdot,\mu)\right\|_{B^{\beta,1}_{1,k}}+\left\|A^{\Theta}(h_1,\rho)(\cdot,\mu)-A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{B^{\beta,1}_{1,k}}.$$

It is evident that

$$A^{\Theta}(h_1,\chi)(u,\mu)-A^{\Theta}(h_1,\rho)(u,\mu)$$
$$=\frac{1}{2\pi b}\, e^{\frac{i}{2b}u(au-\mu)}\int_R h_1(m+u)\,\overline{(\chi-\rho)(m)}\, e^{\frac{i}{b}m(au-\mu)}dm$$
$$=A^{\Theta}(h_1,\chi-\rho)(u,\mu)$$

and similarly

$$A^{\Theta}(h_1,\rho)(u,\mu)-A^{\Theta}(h_2,\rho)(u,\mu)=A^{\Theta}(h_1-h_2,\rho)(u,\mu).$$

Since

$$\int_{[c,\infty[}|(\chi-\rho)(m)|\,(1+C|m|)^N$$
$$\leq \int_{[c,\infty[}|\chi(m)|\,(1+C|m|)^N dm+\int_{[c,\infty[}|\rho(m)|\,(1+C|m|)^N dm$$
$$\leq M_1+M_2,$$

we then write by Theorem 3

$$\left\|A^{\Theta}(h_1,\chi)(\cdot,\mu)-A^{\Theta}(h_1,\rho)(\cdot,\mu)\right\|_{B^{\beta,1}_{1}}$$
$$=\left\|A^{\Theta}(h_1,\chi-\rho)(\cdot,\mu)\right\|_{B^{\beta,1}_{1}}$$
$$\leq\frac{M_1+M_2}{|2\pi b|}\left(\|h_1\|_{B^{\beta,1}_{1,k}}+(K+1)\|h_1\|_{1,k}\right)$$

and

$$\left\|A^{\Theta}(h_1,\rho)(\cdot,\mu)-A^{\Theta}(h_2,\rho)(\cdot,\mu)\right\|_{B^{\beta,1}_{1,k}}$$
$$=\left\|A^{\Theta}(h_1-h_2,\rho)(\cdot,\mu)\right\|_{B^{\beta,1}_{1,k}}$$
$$\leq\frac{M_2}{|2\pi b|}\left(\|h_1-h_2\|_{B^{\beta,1}_{1,k}}+(K+1)\|h_1-h_2\|_{1,k}\right).$$

This completes the proof.

4 Conclusion

In this paper, we have extended the analysis of time-frequency representations by investigating the continuity properties of the linear canonical ambiguity function (LCAF). Unlike traditional approaches that focus on Hardy or BMO spaces, our study establishes that the LCAF serves as a robust tool within the framework of Besov spaces and weighted Besov spaces.The findings demonstrate that the LCAF preserves essential continuity characteristics when mapped onto these spaces, providing a more flexible and refined analysis for non-stationary signals. Furthermore, by deriving the Besov and weighted Besov distances between LCAFs associated with different signals and functionals, we have provided a quantitative measure for signal discrimination and error analysis in the linear canonical domain.

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