On a System with Multiple Period Annuli

ANITA KIRICHUKA[1], FELIX SADYRBAEV[1,2]
[1]Daugavpils University,
13 Vienibas Street, Daugavpils,
LATVIA

[2]Institute of Mathematics and Computer Science,
University of Latvia,
Rainis boulevard 29,
LATVIA

Abstract: - We consider two-point boundary value problems for the Hamiltonian system of the form $x' = f(x,y),\ y' = g(x,y)$, where $f(x,y)$ and $g(x,y)$ are functions with parameters. We estimate the number of positive and oscillatory solutions for the boundary value problems. Our primary tool is the phase plane analysis combined with evaluations of time map functions. Multiple positive solutions are detected due to multiple period annuli.

Key-Words: - ordinary differential equations, oscillation, period annuli, boundary conditions, multiple positive solutions, Hamiltonian systems.

Received: October 24, 2024. Revised: February 9, 2025. Accepted: March 23, 2025. Published: May 5, 2025.

1 Introduction

The theory of boundary value problems (BVP) for ordinary differential equations (ODE) is well-developed in the literature. Many books, [1], [2], [3], [4], [5] mention, among other things, the boundary value problems. The Hamiltonian systems [6], [7] are one of the ODE classes more often considered. Energy-saving arguments are usually involved in treating them. The phase space consideration is also convenient due to the geometrical treatment of trajectories as level sets of Hamiltonians. From a practical point of view, detecting positive solutions is essential. Some links on the subject can be found in [6], [8] and the references therein. The existence of solutions is the main issue when considering the nonlinear BVP for ODE. This stage should precede the computational analysis. The study of multiple solutions to BVP is interesting for theoreticians and essential for practitioners' processes.

We want to address the problem of the existence of families of oscillatory solutions with the existence of solutions to boundary value problems. We restrict ourselves to the case when there are multiple period annuli (continua of periodic solutions) and positive solutions of a BVP are counted.

2 Preliminaries

Autonomous ordinary differential equations can have multiple oscillatory solutions of different types. The second-order scalar ODE also can exhibit such behavior. An example of this is the system

$$x' = f(x,y),\ y' = g(x,y), \tag{1}$$

where

$$f(x,y) = -y\,(y^2 - k^2), \tag{2}$$

$$g(x,y) = x(x\text{^}2 + (1-\lambda)x - \lambda), \tag{3}$$

k and λ are parameters. The phase portrait is shown in Figure 1 for the values $k = 3$ and $\lambda = 3$.

Let us look at this picture. The system has multiple period annuli. Recall that a period annulus is a continuous family of trajectories surrounding one or several critical points. If there is only one critical point of the type center, we will call this family of trajectories the trivial period annulus. If more than one critical point is included, we call this the nontrivial period annulus. We see nine critical points, five centers, and four saddle points. A standard linearization around these points can verify this. We aim to study this relatively complicated behavior of solutions and draw some conclusions about systems with multiple periods of annuli of different types.

First, we analyze the boundary value problems associated with systems of type (1), looking for positive solutions.

In the next section, we formulate the boundary value problems (BVP) and evaluate the number of solutions. For this, we conduct numerical analysis in combination with some theoretical suggestions.

In the third section, we use results and methods concerning the first zero function, similar to those in [6] and [8].

Finally, we discuss the systems with multiple annuli periods and make suggestions.

3 The System

Consider the system (1) with functions (2) and (3)

$$\begin{cases} x' = -y\,(y^2 - k^2), \\ y' = x(x^2 + (1-\lambda)x - \lambda), \end{cases} \tag{4}$$

k and λ are parameters. This is a system that corresponds to the Hamiltonian function:

$$H(x,y) = \frac{x^4}{4} + \frac{1-\lambda}{3}x^3 - \frac{\lambda}{2}x^2 + \frac{y^4}{4} - \frac{k^2}{2}y^2.$$

Hamiltonian systems are essential in the theory of dynamical systems and various fields. Due to their geometric properties, they are used to demonstrate the developed theories. Figure 1 depicts the phase portrait for system (4).

This system was investigated in the work [9], where the authors succeeded in perturbing it by cubic terms so that the perturbed system has thirteen limit cycles. We are interested in different questions. Let us pass to the first one.

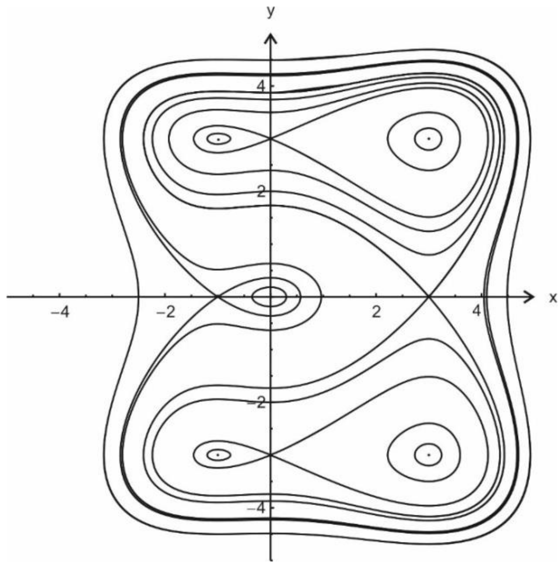

Fig. 1: The phase portrait for system (1) or (4), k=3, $\lambda = 3$

4 The Boundary Value Problem

Consider the system (4) with the boundary conditions:

$$x(0) = 0,\ x(1) = 0. \tag{5}$$

We are looking for positive (and sometimes negative) in x, solutions of the BVP (4), (5).

Due to this system's rich oscillatory behavior, multiple BVP solutions can be expected (4), (5).

We use many results and techniques concerning the first zero function, as shown in [8]. Since we are interested in solutions to problems (4) and (5), we mean by the first zero function (usually also called the time map function) the minimal time needed for $x(t)$, $x(0) = 0$, $y(0) = b$, to vanish again.

First, let us take the test. Consider the system (4) together with the initial conditions:

$$x(0) = 0, \qquad y(0) = b, \tag{6}$$

where the parameter b is in the interval [-5, 5].

We solve this problem numerically, and the calculation results are shown in Figure 2.

Figure 2 shows that the first zero $t_1(b)$ of the solution $x(t)$ crosses the level $t = 1$ at least *12* times. Therefore, at least 6 solutions of the BVP (4),

(5) with positive $x(t)$, and *6* solutions with negative $x(t)$ exist.

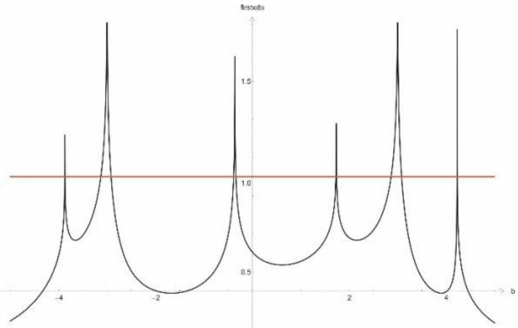

Fig. 2: The graph of the first zero function of the system (4) against the initial value ***b***

The respective values of b are computed and summarized in Table 1.

Table 1. The initial data in (6)

b	value
b_1	-3.8754
b_2	-3.8699
b_3	-3.1299
b_4	-2.9169
b_5	-0.3868
b_6	-0.3398
b_7	1.7236
b_8	1.7389
b_9	2.8642
b_{10}	3.0808
b_{11}	4.2249
b_{12}	4.2307

The trajectory parameterized by $t \in [0,1]$ and corresponding to $y(0) = b_1$ is depicted in Figure 3.

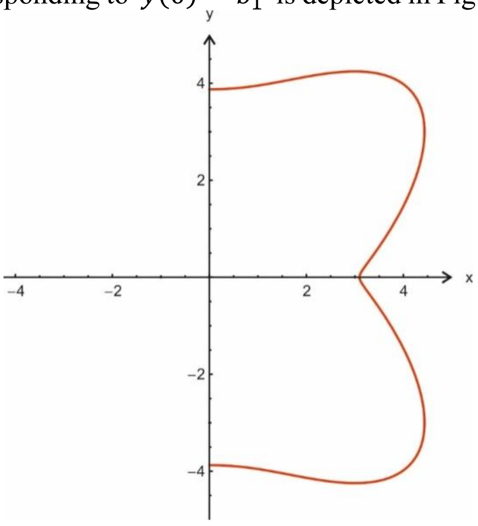

Fig. 3: Phase trajectory on the interval $[0,1]$ with the initial values $x(0) = 0,\ y(0) = b_1$

The x - component and y-component of the solution $(x(t), y(t))$ associated with the trajectory in Figure 3 are depicted in Figure 4 and Figure 5.

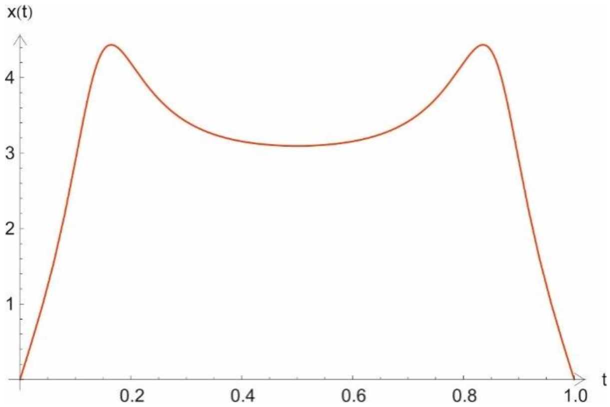

Fig. 4: X-component of a solution

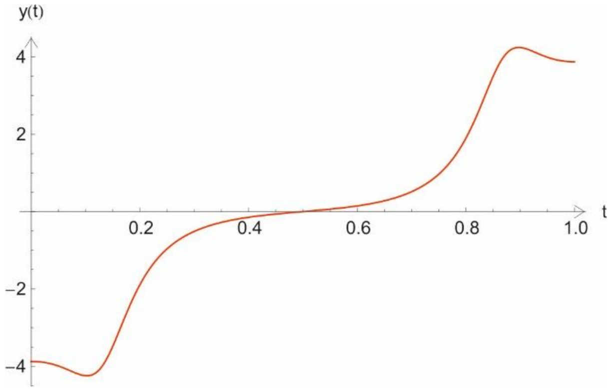

Fig. 5: Y-component of a solution

The first group of trajectories on the interval [0,1], where $y(0)$ from b_1 to b_6, are presented in Figure 6.

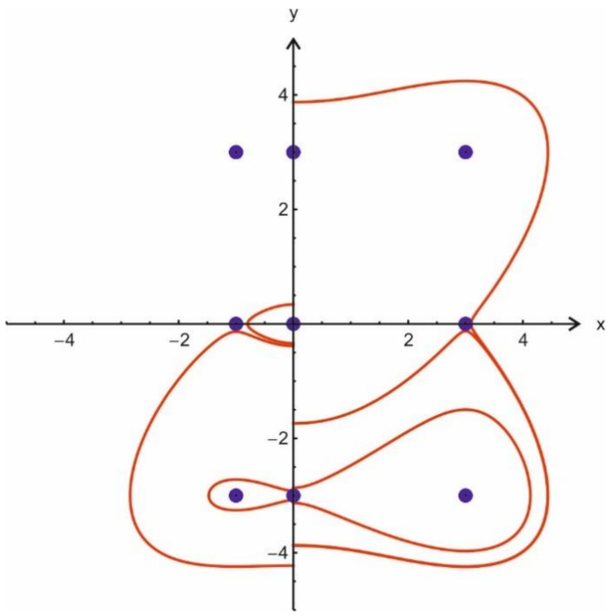

Fig. 6: The first group of trajectories (y(0) changing from b_1 to b_6)

The next group of trajectories on the interval [0,1], where y(0) takes values from b_7 to b_{12}, is presented in Figure 7.

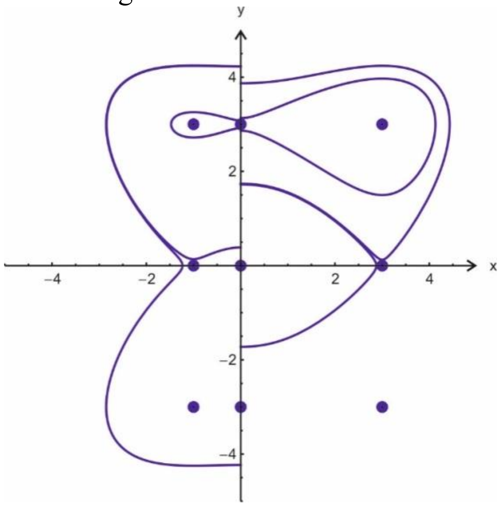

Fig. 7: The group of trajectories (y(0) from b_7 to b_{12})

5 Linearization at Critical Points

System (4) has 9 critical points, 5 of which are the centre points: (0,0), (λ, k), $(\lambda, -k)$, $(-1, k)$, $(-1, -k)$ and 4 are the saddle: $(-1,0)$, $(\lambda, 0)$, $(0, k)$, $(0, -k)$. In this section, we investigate three critical points located at the x=0 axis and show that they are the first $(-1,0)$, the third $(\lambda, 0)$ saddle points, and the middle point (0,0) a center.

Linearization of the system (4) at a critical point (x^*, y^*) yields:

$$\begin{cases} u' = (-3y^{*2} + k^2)v \\ v' = (3x^{*2} + 2(1-\lambda)x^* - \lambda)u. \end{cases} \quad (7)$$

Consider the linearized system at the critical point (0,0)

$$\begin{cases} u' = k^2 v \\ v' = -\lambda u. \end{cases} \quad (8)$$

The eigenvalues of the linearized system (6) are $\mu_{1,2} = \pm k\lambda i$, where i is an imaginary unity.

The system (6) can be rewritten in the form

$$u'' = -k^2 \lambda u \quad (9)$$

And there is a solution

$$u(t) = \sin k\sqrt{\lambda} t, \quad (10)$$

That satisfies the initial condition $u(0) = 0$, $u'(0) \neq 0$.

Appropriate solutions $(u(t), v(t))$ of the linear system (7) provide approximations to solutions of the Cauchy problems (4), $x(0) = 0$, $y(0) = \pm\varepsilon$, where $\varepsilon > 0$ is a small value. We suppose that ε is so small that $\varepsilon < \alpha_2$ where $(0, \alpha_2)$ is a point of intersection of the "upper" heteroclinic solution with the y-axis. Also, $\varepsilon > \alpha_1$ where $(0, \alpha_1)$ is a point of intersection of the "lower" heteroclinic solution with the y -axis. Two heteroclinic trajectories connect two saddle points $(-1,0), (\lambda, 0)$.

Therefore, if $t(\pm\varepsilon)$ is the time needed for a point to move along the phase trajectory of the system (4) from the point $(0, \pm\varepsilon)$ to $(0, \mp\varepsilon)$ then $t(\pm\varepsilon) \approx \frac{\tau}{2} = \frac{\pi}{k\sqrt{\lambda}}$, where $\tau = \frac{2\pi}{k\sqrt{\lambda}}$ is the period of solution (10). This is true for ε close to zero.

Consider solutions of system (4) and conditions $x(0) = 0$, $y(0) = \varepsilon$, where ε close to zero. Then $x(t)$ has exactly n zeros in the interval (0,1) (and $x(1) \neq 0$) if the inequalities $\frac{\pi n}{k\sqrt{\lambda}} < 1 < \frac{\pi(n+1)}{k\sqrt{\lambda}}$ hold. On the other hand, if $\varepsilon \to \alpha_2$, then the respective $x(t)$ has no zeros in the interval (0,1].

Therefore, by continuity arguments, there are at least n solutions to the problem (4), (5). Considering solutions of (4) with the initial values $x(0) = 0$, $y(0) = -\varepsilon$, where ε changes from zero to $-\alpha_1$, we get additional at least n solutions. Hence, at least $2n$ solutions with the trajectories in the central trivial annulus.

Continue the analysis of critical points. They are nine, with $x \in \{-1, 0, \lambda\}$, $y \in \{-k, 0, k\}$. The matrix of coefficients of the linearized system (7) is

$$A = \begin{pmatrix} 0 & -3y^{*2} + k^2 \\ 3x^{*2} + 2(1-\lambda)x^* - \lambda & 0 \end{pmatrix}$$

and the respective characteristic equation (concerning μ) $\det(A - \mu E) = 0$, where E is the unity matrix, is:

$$\mu^2 = (k^2 - 3y^{*2})(3x^{*2} + 2(1-\lambda)x^* - \lambda) \quad (11)$$

Recall that (x^*, y^*) are the coordinates of the critical point in question.

a) Consider the critical points at $(0, k)$ and $(0, -k)$. The characteristic equation (11) takes the form $\mu^2 = 2\lambda$, and the critical points are saddles.
b) Consider the critical points at $(-1, -k)$, $(-1, k), (-1, 0)$. The characteristic equation (11) takes the form $\mu^2 = -2k^2(1+\lambda)$, and the characteristic numbers μ_1 and μ_2 are imaginary conjugates. The critical points $(-1, -k), (-1, k)$ are centers. This is not the case for the critical point $(-1, 0)$. The characteristic equation is $\mu^2 = k^2(1+\lambda)$, and the critical point $(-1, 0)$ is a saddle.
c) Consider the critical points at $(\lambda, -k)$, (λ, k), $(\lambda, 0)$. The characteristic equation (11) takes the form $\mu^2 = -2k^2\lambda(1+\lambda)$, and the points $(\lambda, -k)$, (λ, k) are centers. The characteristic equation for the point $(\lambda\ 0)$is $\mu^2 = k^2\lambda(1+\lambda)$, and this point is a saddle.

This analysis is in agreement with the phase portrait in Figure 1.

6 The Period Annuli

In the phase portrait depicted in Figure 1, we see several period annuli. There are five trivial period annuli surrounding the centers at $(-1, -k)$, $(-1, k)$, $(0,0)$, $(\lambda, -k)$, $(\lambda, -k)$. Only in the third annulus can a positive solution of the BVP exist. Does it exist depends on the result of the linearization around the origin. The condition for the existence is provided in Section 4.

There are also four nontrivial period annuli containing more than one critical point (more details about nontrivial period annuli in [2]). Two of them are symmetric and include the groups of critical points with $y = k$ and $y = -k$. Let us denote them PAU (period annulus upper) and PAL (period annulus lower). The outer trajectories in both period annuli pass close to the saddle point at $(\lambda, 0)$. Therefore, they have very large (tending to infinity) periods. The inner trajectories in PAU and PAL pass by the saddles at $(0, k)$ and $(0, -k)$, respectively. Their periods are unbounded also. Therefore, the trajectory with the minimal period exists in any of these annuli. If the time of the "positive half-period" is less than 1, then a positive solution of the BVP (4), (5) exists in both period annuli.

The period annulus is also topologically equivalent to Figure 8, with the central point at $(\lambda, 0)$. It also has outer and inner trajectories with arbitrarily large periods. Therefore, the situation is similar to the above described, and the existence of a positive solution of BVP depends on the minimal "positive half-period". Moreover, an outer unbounded period annulus exists, containing all critical points and is potentially suitable for the existence of a positive solution.

For the values $k = 3$, $\lambda = 3$ Figure 3 (corresponding to $y(0) = b_1$), Figure 8 (corresponding to $y(0) = b_3; b_7; b_9$) and Figure 9 (corresponding to $y(0) = b_2; b_8$) shows this positive solution.

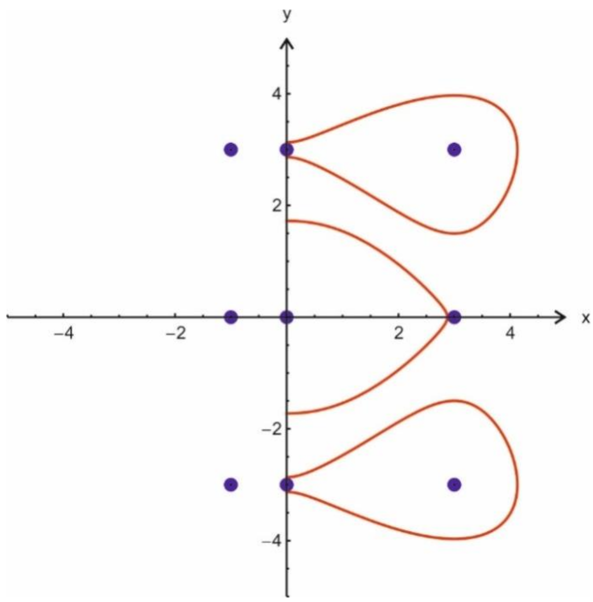

Fig. 8: Phase trajectory on the interval [0,1] with the initial values $\mathrm{x}(0) = 0$, $\mathrm{y}(0) = \mathrm{b}_3; \mathrm{b}_7; \mathrm{b}_9$

The x-components of solutions with the initial values $x(0) = 0$, $y(0) = b_1; b_3; b_7; b_9$ and $x(0) = 0$, $y(0) = b_2; b_8$ are depicted in Figure 10 and Figure 11.

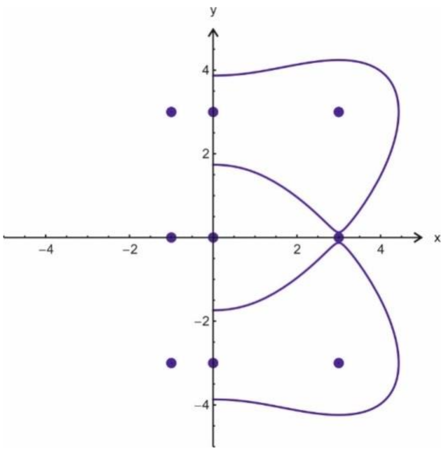

Fig. 9: Phase trajectory on the interval $[0,1]$ with the initial values $x(0) = 0,\ y(0) = b_2; b_8$

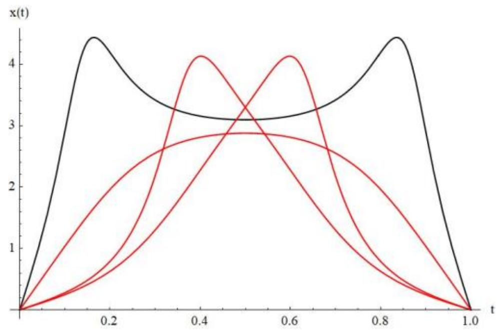

Fig. 10: x-component of a solution with the initial values $x(0) = 0,\ y(0) = b_1; b_3; b_7; b_9$

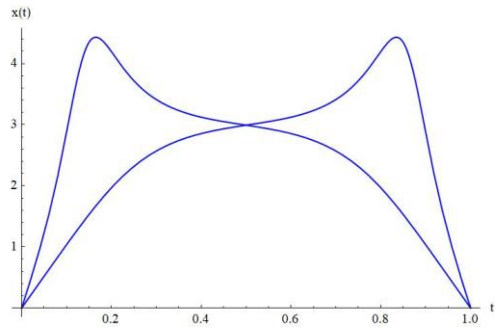

Fig. 11: x-component of a solution with the initial values $x(0) = 0,\ y(0) = b_2; b_8$

7 Solutions to the Boundary Value Problem

Positive solutions of the BVP in question are detected numerically, but analytical proof can also be given in the spirit of [6], [8], [10]. The results of the computational study are depicted in Figure 8, Figure 9, Figure 10 and Figure 11.

In Figure 8, three trajectories corresponding to the positive solutions of the BVP are depicted. One has that $x(0) = x(1) = 0$ and as a byproduct, $y(0) = b_3$, $y(0) = b_7$, $y(0) = b_9$ (the values of b are in Table 1).

In Figure 9, two trajectories of the nontrivial period annuli PAU and PAL (section 6) corresponding to the positive solutions of the BVP are depicted. One has that $x(0) = x(1) = 0;$ *the* additional information is that $y(0) = b_2,\ \ y(0) = b_8$. The one big trajectory belonging to the outer nontrivial period annulus and associated with the positive solution of the BVP is depicted in Figure 3. Therefore, the following assertion is true.

Proposition. The boundary value problem (4), (5), where $k = 3$ and $\lambda = 3$, has six positive solutions. The respective segments of trajectories belong to six period annuli, of which three are trivial, and the remaining three are nontrivial.

8 More on Figure 1

The Hamiltonian $H(x, y) = \frac{x^4}{4} + \frac{1-\lambda}{3}x^3 - \frac{\lambda}{2}x^2 + \frac{y^4}{4} - \frac{k^2}{2}y^2$ and the horizontal plane H=-11.3 are depicted in Figure 12 and Figure 13.

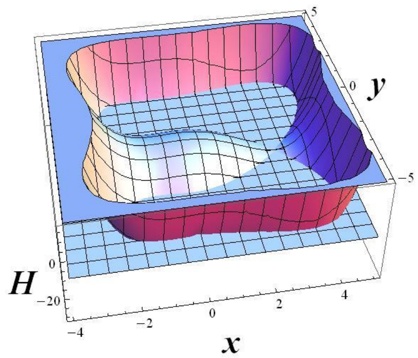

Fig. 12: Hamiltonian. Top view

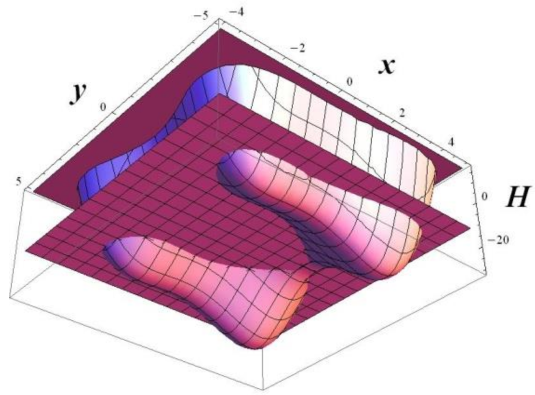

Fig. 13: Hamiltonian. Bottom view

The level sets of Hamiltonian when combined form the phase portrait of the system under consideration.

The minima of Hamiltonian can be observed in Figure 13. They correspond to the critical points of the type “center” (four of them). From these pictures, one can conclude that the center points corresponding to minima in the regions PAU and PAL are stable with respect to small perturbations. In these regions the graph of Hamiltonian has wells. The remaining center point at the origin which is seen in Figure 12 is unstable. Figure 14 shows the character of movement along the trajectories.

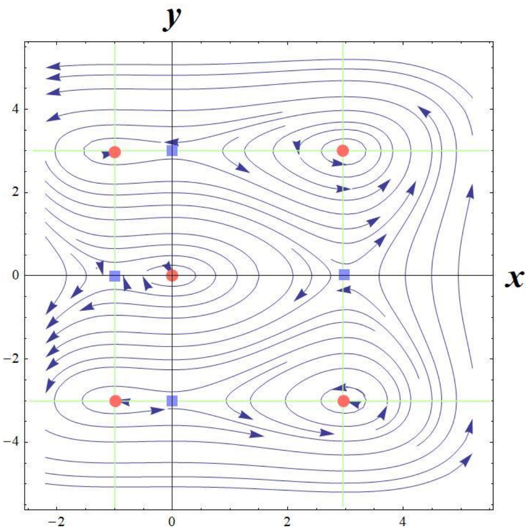

Fig. 14: The vector field for the system (4), $k = 3$ and $\lambda = 3$, saddles and centers marked

9 Conclusion

A class of nonlinear oscillators exhibiting various oscillatory behaviors is considered. Systems in this class contain period annuli, which are related to each other [8]. The properties of these period annuli significantly affect the number of solutions satisfying certain additional conditions, such as boundary conditions and the positivity condition. Motivating by an example presented in [9], we evaluated the number of positive solutions of the BVP (4), and (5) and provided exhaustive information about them. If the parameters k and λ vary so that topologically the phase portrait is similar, the trajectories going from the vertical axis to it again and lying in the right half-plane are saved, but the number of solutions to BVP may change. Computational analysis is needed.

The proposed approach is suitable for systems of the form:

$$\begin{cases} x' = -y\,P(y), \\ y' = \;\; x\,Q(x), \end{cases} \tag{12}$$

where P and Q are polynomials.

Studying various coupled oscillators' interactions remains an essential problem in the theory of ordinary differential equations.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

All authors have contributed equally to creation of this article.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors declare no conflict of interest.