A Novel Approach to Modelling Overdamped Second and Higher-Order Systems using Linearized Derived Equations

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Abstract: - A second-order system is widely recognized in control systems, as many practical systems are modeled using it. The system's response to a step input is well-defined, with mathematical expressions available for key parameters like rise time, settling time, and others. However, most of these equations apply specifically to an underdamped second-order system, where an explicit solution is relatively straightforward, except for the delay time equation, which is derived from a linear equation involving the damping factor, ς. This paper develops mathematical equations for both delay time and rise time-based on linear equations, allowing the extraction of a mathematical model from the system's output response for both low and high damping factor values. Additionally, the proposed equations can be applied to model higher-order systems by using an equivalent second-order system, with results showing that this model accurately represents the higher-order system. Further analysis investigates the effect of the damping factor on natural frequency ratio (ς/w_n) at high ς values, demonstrating that the system's response depends on ς/w_n rather than the individual values of ς or w_n. This implies that the system response remains consistent for a fixed ς/w_n ratio.

Key-Words: - over damped second order system, delay time, rise time, natural frequency ratio, damping ratio, mathematical model, higher order system.

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1 Introduction

Second-order systems are prevalent in various fields of engineering, particularly in control systems, where their dynamic behavior provides insight into stability, performance, and response characteristics. These systems are generally represented by a second-order differential equation, characterized by two main parameters: the natural frequency (ω_n) and the damping ratio (ζ). These parameters govern how the system responds to inputs, which can range from oscillatory behavior in underdamped systems (ζ<1) to critically damped (ζ=1) or overdamped responses (ζ>1) that exhibit slower. Non-oscillatory behavior response of a second-order system is classified based on how it reacts to step inputs, and this is a concept central to control theory. This classification enables engineers to predict, design, and control system behaviors effectively. Understanding second-order systems' response characteristics—such as peak time, overshoot, settling time, and steady-state error—is crucial for optimizing these applications and ensuring system stability and robustness, [1].

Delay time (t_d) and settling time(t_s) are essential parameters in analyzing and designing second-order system. Where Delay time is the time it takes for the system's response to reach a specified fraction of the final value after a step input, typically 50% of the steady-state value [2], [3], [4], and the settling time [t_s] is the time it takes for the system's response to remain within a specified percentage (commonly 2% or 5%) of its final steady-state value after a step input, [2], [3], [4].

The formulas for delay time, settling time, and rise time are specifically derived for underdamped systems. These formulas are useful for establishing a mathematical model and solution for systems whose output behavior resembles that of an underdamped response. However, for overdamped second-order systems, the absence of precise formulas for delay time and settling time makes it challenging to directly determine key parameters such as the damping ratio (ζ) and natural frequency (ω_n) from the system's output, [5], [6], [7].

Most of the previous methods such as Vítečková's method, Latzel's method, Harriott's method, Smith's method, Strejc's method and Sundaresan's & Krishnaswamy's method, are used to model the overdamped system but none of them define equations for the delay and settling time of the system, [5], [6], [7], [8], [9], [10].

This paper addresses this gap by deriving mathematical formulas through numerical solutions of the second-order equation, accompanied by an in-depth investigation of these formulas.

2 Over damped Second Order System

For the second-order system:

$$T(s)=\frac{w_n^2}{s^2+2\varsigma w_n s+w_n^2} \qquad (1)$$

When $\varsigma > 1$, the system has two real distinct poles, and the system is called an overdamped system.

The solution to the over damped system is:

$$y(t)=1+\frac{w_n}{2\sqrt{\varsigma^2-1}}\left(\frac{e^{-s_1 t}}{s_1}-\frac{e^{-s_2 t}}{s_2}\right) \qquad (2)$$

Where:

$$s_{1,2}=\varsigma w_n \pm w_n\sqrt{\varsigma^2-1} \qquad (3)$$

The delay time t_d is defined as the time required to reach 50% of the final value. t_d is calculated by substitute $y(t_d) = 0.5$ as follows:

$$0.5=1+\frac{w_n}{2\sqrt{\varsigma^2-1}}\left(\frac{e^{-s_1 t_d}}{s_1}-\frac{e^{-s_2 t_d}}{s_2}\right) \qquad (4)$$

There is no explicit solution for the above equation, but it can be solved using numerical techniques such as Newton Raphson method, before solving equ.(4) it can be simplified as follows:

$$0.5=1+\frac{1}{2\sqrt{\varsigma^2-1}}\left(\frac{e^{-s_1 t_d}}{z_1}-\frac{e^{-s_2 t_d}}{z_2}\right) \qquad (5)$$

where : $z_1=\varsigma+\sqrt{\varsigma^2-1}$

$z_2=\varsigma-\sqrt{\varsigma^2-1}$

This simplification reduces the number of variables, where it can be solved for the variable ($w_n t_d$) for a given value of ς.

A simple code using Newton Raphson method can be used to solve the equation as follow:

$$f(w_n t_d)=\frac{1}{\sqrt{\varsigma^2-1}}\left(-\frac{e^{-s_1 t_d}}{z_1}+\frac{e^{-s_2 t_d}}{z_2}\right) \qquad (6)$$

$$\frac{d}{d(w_n t_d)}f(w_n t_d)=\frac{1}{2\sqrt{\varsigma^2-1}}\left(e^{-s_1 t_d}-e^{-s_2 t_d}\right) \qquad (7)$$

$$\Delta(w_n t_d)=\frac{c-f(w_n t_d)}{\frac{df(w_n t_d)}{d(w_n t_d)}} \qquad (8)$$

where c =;1 (i.e $2*(1-y(t_d))=2*0.5=1$)

Settling time is defined as the time when the output reaches 95% or 98% of the final value for the first time. To find the settling time (t_s) it is required to find the time when the output equal 0.98 from the final value, and this also leads to a nonlinear equation, which requires numerical solver to find its value.

The same equation from equ.(6) to equ.(8) except that $c = 2*(1-y(t_s))=2*(1-0.98)=0.04$.

3 Results

Varying ς from 1 to 5 in step of 0.01,

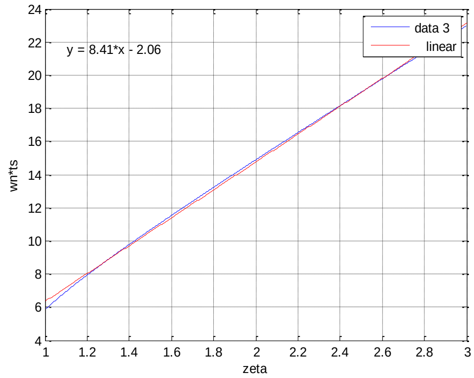

Fig. 1: Relation between ς and ($w_n t_s$) for low values of ς

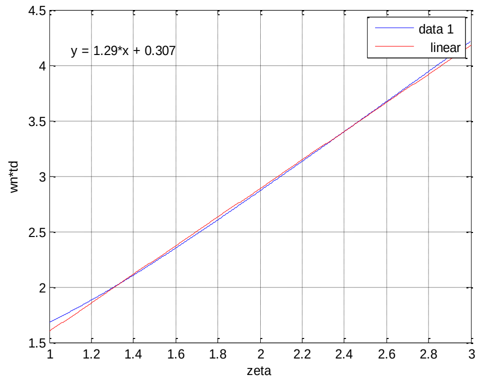

Fig. 2: Relation between ς and ($w_n t_d$) for low values of ς

It is clear from the above Figure 1 that the relation between $w_n t_d$ and ς is linear and it can be given as follows:

$$w_n t_d = 1.29\varsigma + 0.309 \tag{9}$$

then $$t_d = \frac{1.29\varsigma + 0.309}{w_n} \tag{10}$$

Figure 2 also shows a linear relation between ς and $w_n t_s$, and the linear equation is given by:

$$w_n t_s = 8.41\varsigma - 2.06 \tag{11}$$

The settling time (t_s) can be defined as:

$$t_s = \frac{8.41\varsigma - 2.06}{w_n} \tag{12}$$

For larger values of ς, $\varsigma > 5$

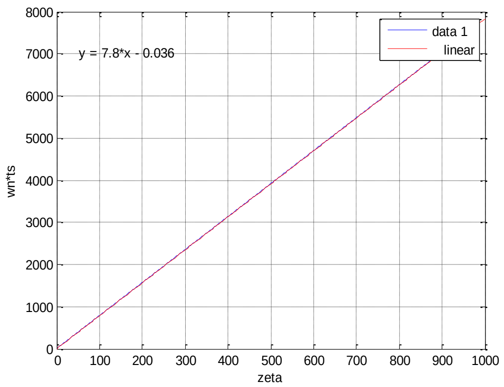

Fig. 3: Relation between ς and ($w_n t_s$) for large values of ς

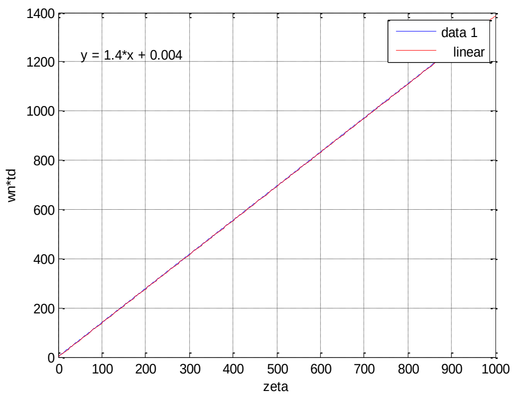

Fig. 4: Relation between ς and ($w_n t_d$) for large values of ς

Based on Figure 3 and Figure 4, the relation between the delay time (t_d) and ς is linear and between settling time(t_s) and ς is also linear and can be given by :

$$t_d = \frac{1.4\varsigma}{w_n} \tag{13}$$

and

$$t_s = \frac{7.8\varsigma}{w_n} \tag{14}$$

Divide equ.(14) by equ.(13) yields to:

$$t_s = \frac{7.8}{1.4} t_d = 5.5714 t_d \tag{15}$$

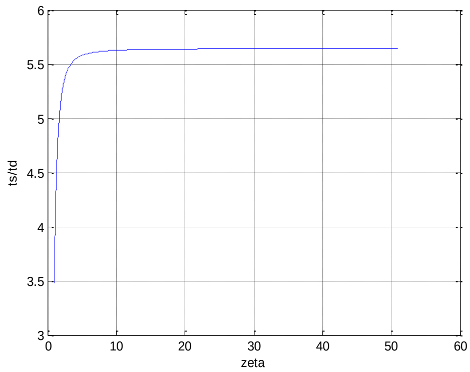

Fig. 5: Relation between ς and (t_s/t_d)

Figure 5, shows that the relation between the settling time and the delay time is constant for large value of ς, and this ratio is the same as the value-driven in equ.(15) (i.e 5,5714).

4 Validate the Results

To prove the derived mathematical equation for the delay time and the settling time,

$$T_1(s) = \frac{25}{s^2 + 20s + 25} \tag{16}$$

for this system $w_n = 5$ and $\varsigma = 2$, the response of the system is shown in Figure 6.

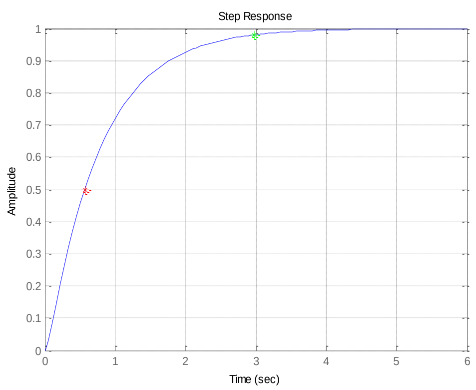

Fig. 6: Output response for unit step input for the system $T_1(s)$ in equ.(16)

From Figure 6 the delay and settling time are 0.5732s, 2.9761s respectively.

Using equ.(10) and equ.(12) the values of t_d and t_s are:

$$t_d = \frac{1.29(2)+0.309}{5} = 0.5778s \qquad (17)$$

and

$$t_s = \frac{8.41(2)-2.06}{5} = 2.952s \qquad (18)$$

for large value of ς, (i.e $\varsigma > 5$), let

$$T_2(s) = \frac{25}{s^2+100s+25} \qquad (19)$$

The response of the system for a unit step function is shown in Figure 7. From the figure the delay time and settling time are 2.7811s and 15.62s, respectively.

First, the ratio between these values is:

$$15.62/2.7811 = 5.616$$

using eu.(10) and equ.(2) :

$$t_d = \frac{1.4(10)}{5} = 2.8s \qquad (20)$$

and

$$t_s = \frac{7.8(10)}{5} = 15.6s \qquad (21)$$

Table 1 shows the values of the delay time t_d and settling time t_s extracted from the response of the transfer function and delay time t_d' and settling time t_s' using the mathematical equations. Relative error of delay time and settling time are shown in Table 1, where the error doesn't exceed 1.4% and for large value of ς it approaches to 0%.

Since the damping factor ς is determined the equation that can be used to evaluate delay time and settling time, it is worth showing the effect of w_n in these equations.

Equ.(10) and equ.(12) are used to calculate the delay time and settling time for low value of ς (i.e $\varsigma < 5$), for all values of w_n. Figure 7 shows that the relative error for both of delay time and settling time is less than 0.001 for ς =2. Equ.(13) and equ.(14) are used for $\varsigma > 5$ regardless the value of w_n. Value of $w_n$is varied in Figure 8, the relative error doesn't exceed 0.001 for $\varsigma = 20$.

Table 1. Settling time (t_s), delay time (t_d) extracted from output response, calculated time delay (t_d') and calculated settling time (t_s') for several values of ς

ς	t_d [s]	t_d' [s]	t_s [s]	t_s'' [s]	t_d relative error [%]	t_s relative error [%]
2.0000	0.5700	0.5778	2.9800	2.9800	1.3684	0.9396
2.5000	..7050	.7068	.8000	3.8000	0.6953	0.4719
3.0000	.8400	.8358	.6000	4.6000	0.6051	0.5990
3.5000	0.9800	.9648	.4000	5.4000	0.9702	0.9118
4.0000	.1150	.0938	6.1950	6.1950	1.2805	1.2686
35.0000	.7250	.7230	54.835	54.8350	0.0684	0.0570
40.0000	.1150	11.112	62.670	2.6700	0.0453	0.0306
45.0000	.5050	12.501	70.505	70.5050	0.0398	0.0211
50.0000	.8900	13.890	78.345	78.3450	0.0264	0.0140
55.0000	.2800	15.279	86.175	86.1750	0.0206	0.0114
60.0000	16.670	16.668	94.010	94.0100	0.0184	0.0096
65.0000	18.060	18.057	01.845	01.845	0.0180	0.0082
70.0000	19.450	19.446	09.680	109.680	0.0185	0.0070
75.0000	20.835	20.835	17.520	117.520	0.0149	0.0061
80.0000	22.225	22.224	25.350	125.350	0.0130	0.0053
85.0000	23.615	23.613	33.185	133.185	0.0122	0.0045
90.0000	25.005	25.002	41.020	141.020	0.0122	0.0039
95.0000	26.390	26.391	48.855	148.855	0.0109	0.0034
100.00	27.78	27.78	56.69	156.690	0.0094	0.0029

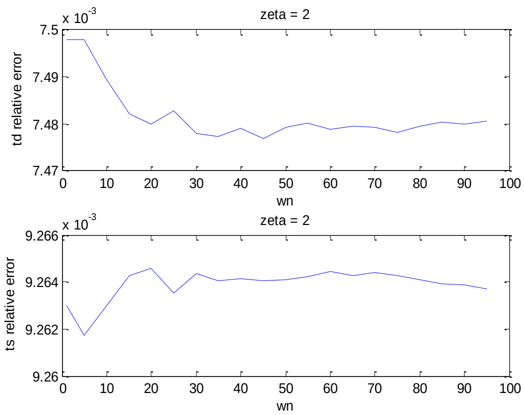

Fig. 7: Relative error of td and t_s for several values of w_n for ς=2

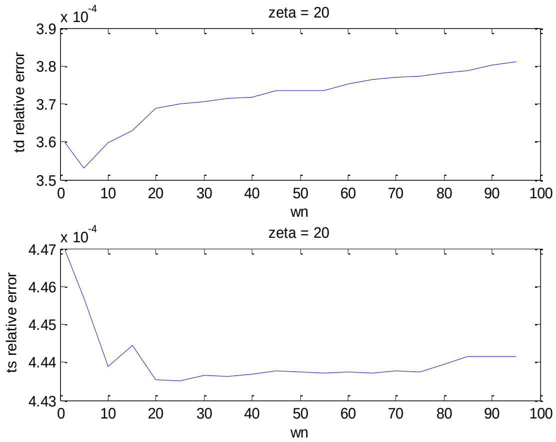

Fig. 8: Relative error of td and ts for several values of w_n for ς=20

5 Extract the Second-Order Equation from the Output Response

A lot of systems can't be modeled mathematically, while its output response can be plotted, based on the output, a linearized model can be extracted from the output. As mentioned earlier, that over damped system response doesn't have an explicit equation to derive the value of ς and w_n from them.

The derived mathematical equation will be used here to get the linear model, first it will be extracted from a known second-order system to compare the results, then from a known higher-order system.

5.1 Second Order System

$$\text{for } T(s) = \frac{100}{s^2 + 40s + 100} \qquad (22)$$

ς=2 and w_n =10,
the response of the system is shown in Figure 9.

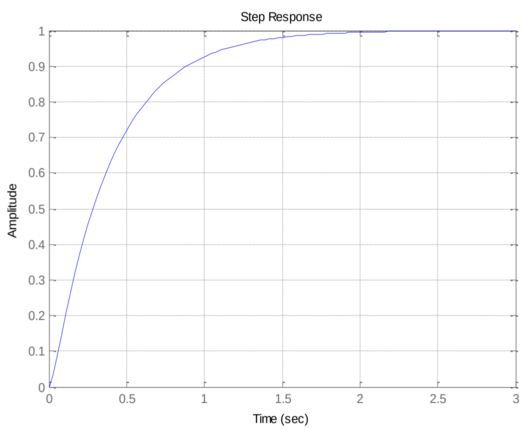

Fig. 9: Output response for unit step input for the system T(s) in equ.(22)

From Figure 9 t_d=0.2866 and t_s=1.488. To extract the values of ς and w_n from these values, we need to know the range of ς before determining which equation must be used, but this is impossible since the value of ς is unknown, so we can use the value of t_s/t_d instead, as this value reaches the value of 5.571 then ς consider high, otherwise ς will be considered low.

For the given values of t_d and t_s, the value of t_s/t_d = 5.192, which is less than the expected value for high value of ς(i.e 5.571), then the equ.(10) and equ.(12) can be used.

Divideequ.(12) by equ.(10), then

$$5.1916 = \frac{8.41\varsigma - 2.06}{1.29\varsigma + 0.309} \text{ then } \varsigma = 2.1$$

Substitute this value in either equ.(10) or equ.(12).

$$0.287 = \frac{1.29(2.1) + 0.309}{w_n} \text{ , then } w_n = 10.5$$

Then the approximate mathematical model is:

$$T_1(s) = \frac{110.25}{s^2 + 44.1s + 110.25} \qquad (23)$$

The difference between the output from equ.(22) and equ.(23) is shown in Figure 10.

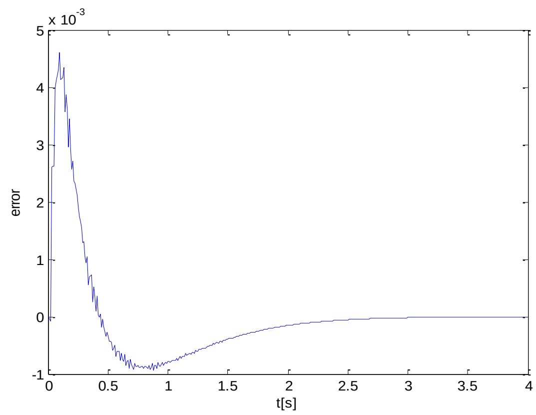

Fig. 10: Difference between the output and the proposed output of T(s) in equ.(22)

As it is shown in Figure 10, the difference between the two output is very small and less than 0.005.

6 Extract the Second Order Equation for Higher Order System

For higher order system, for example third order system, if the third pole is located closer to jw-axis than the poles of the second order system, then the system can't be simplified by a second-order system. Since the response of the system can't distinguish between second-order or higher-order, it is hard to tell from the plot the order of the system.

To investigate this point of the proposed mathematical model, several locations of the third pole will be assumed based on the locations of the second-order poles.

Lets us continue with T(s) described in equ.(22), the locations of the poles are :

$$s_1 = -\varsigma w_n - w_n\sqrt{\varsigma^2 - 1} = -37.32$$

$$s_2 = -\varsigma w_n + w_n\sqrt{\varsigma^2 - 1} = -2.6795$$

several locations of the third pole s_3 will be chosen"

a. s_3 near to jw-axis more than the two poles.

For this purpose, let assume s_3 = -0.4, then the response of the system shown in Figure 11.

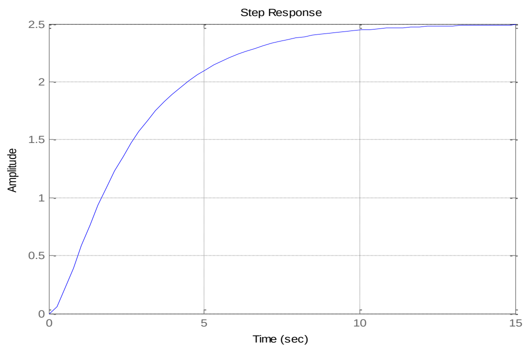

Fig. 11: Output response for unit step input for the system T(s) in equ.(22) after adding the third pole (s_3=-0.4)

From Figure 11, t_d = 2.1635s, t_s = 10.2275s. Using equ.(12) and equ.(10) , then the values of ς = 1.517 and w_n = 0.9549. These values are far away from ς and $w_n$of T(s), which are ς =2 and w_n = 10.

But this makes sense, since the two nearest poles to jw-axis are s_2 and $s_3$and they will be considered the two poles of the second-order system that determine the system response. Based on these poles, the value of ς and w_n can be calculated as follows:

$$w_n = \sqrt{s_2 s_3} = \sqrt{-0.4(-2.6795)} = 1.03$$

$$\varsigma = \frac{-(s_2 + s_3)}{2w_n} = \frac{-(-0.4 - 2.6795)}{2(1.03)} = 1.495$$

Note that these values are conceded with the values derived from the system response.

Figure 12 shows the response of the third-order system and the response of the derived second-order system (i.e ς = 1.517 and w_n = 0.9549), where it is difficult to distinguish between the two curves.

Figure 13 shows the difference between the two outputs, the error doesn't exceed 0.002.

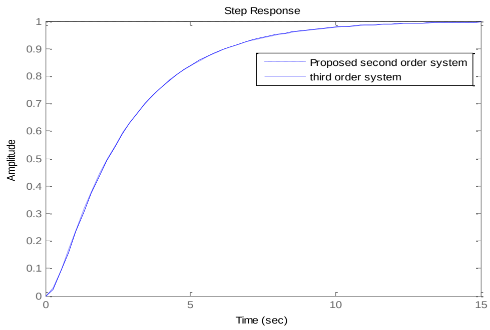

Fig. 12: Response of the third-order system (s_3=-0.4) and the response of the derived second-order system ς = 1.517 and w_n = 0.9549)

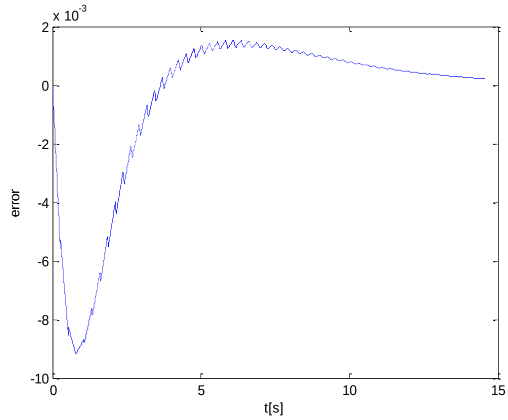

Fig. 13: Difference between the output of the third order system (s_3=-0.4) and the output of the derived second order system ς = 1.517 and w_n = 0.9549)

b. Locate the third pole between s_1 and s_2.

Let s_3 = -10. the response of the third-order system is shown in Figure 14.

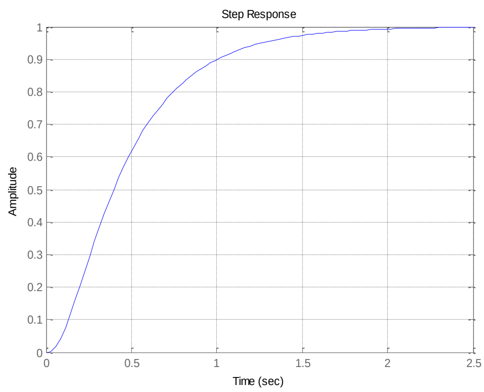

Fig. 14: Output response for unit step input for the system T(s) in equ.(22) after adding the third pole (s_3= -10)

From Figure 14, the values of t_d = 0.396s and t_s = 1.606s.

Using these values to calculate the values of ς and w_n by equ.(10) and equ.(12) ς = 1.042 and w_n = 4.176.

The dominant poles of the third-order system is s_2 and s_3,by using these two poles to determine the values of ς and w_n as follows :

$$w_n = \sqrt{s_2 s_3} = \sqrt{-10(-2.6795)} = 5.17$$

$$\varsigma = \frac{-(s_2+s_3)}{2w_n} = \frac{-(-10-2.6795)}{2(5.17)} = 1.2263$$

Note that the values of ς and w_n are not close. The response of the third-order system, the proposed second-order system (i.e ς = 1.042 and w_n= 4.176) and the reduced second-order system (i.e ς = 1.2263 and w_n = 5.17) are shown in Figure 15.

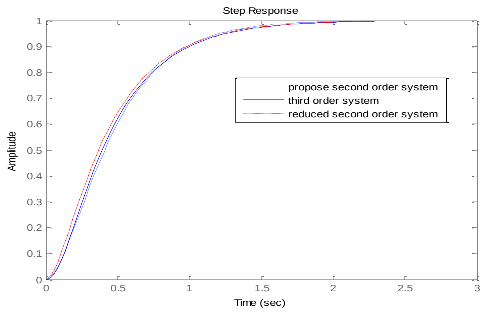

Fig.15 Response of the third-order system (s_3=-10), the proposed second-order system (ς = 1.042 and w_n = 4.176) and the reduced second-order system (ς = 1.2263 and w_n = 5.17)

In Figure 15, the proposed second-order system is closer to the third-order system than the reduced second-order system.

c. Locate the third pole to the left of s_1.

Let s_1 = -50; the response of the third-order system is shown in Figure 16.

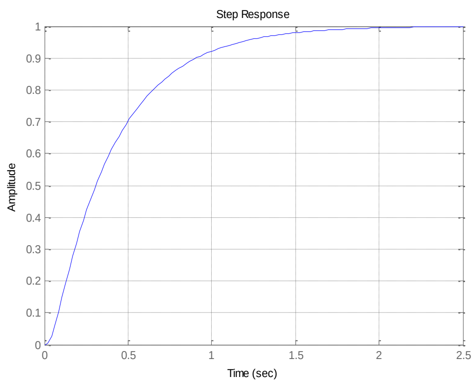

Fig. 16: Output response for unit step input for the system T(s) in equ.(22) after adding the third pole (s_3= -50)

From the curve in Figure 16, the values of t_d=0.3070sand t_s = 1.5100s.

Using these values to calculate the values of ς and w_n by equ.(10) and equ.(12) ς = 1.73 and w_n = 8.29.

The difference between the response of the third-order system and the proposed second-order system is shown in Figure 17.

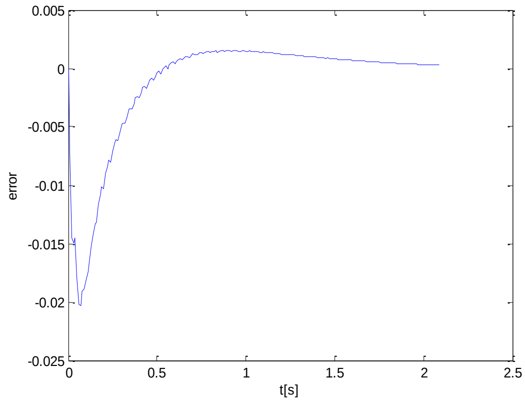

Fig. 17: Difference between the response of the third order system (s_3=-50) and the proposed second order system (ς = 1.73 and w_n = 8.29)

For large values of ς , the ratio between t_s and t_d is constant, which means that the two equations of both of t_d and t_s are reduced to one equation and will be useful to get the value of ς/w_n.

To examine the effect of $\varsigma/w_n$for a large value of ς, let ς > 20, and ς/w_n=2, the response of several systems that have several values of ς and w_n such that ς > 20 and ς/w_n =2 are shown in Figure 18 and Figure 19. The main reason to plot system response in separate subplot because it is hard to distinguish between them if they are plotted at the same plot.

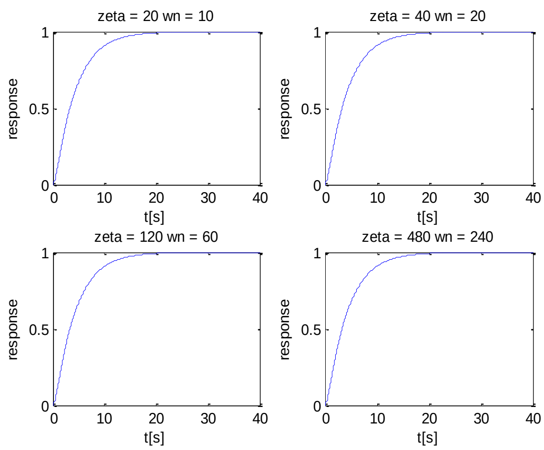

Fig. 18: Output response for unit step input for the second-order system for ς/w_n=2 with different values of ς and w_n

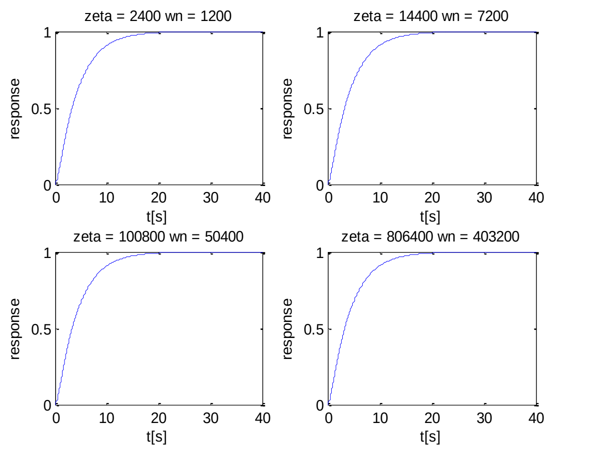

Fig. 19: Output response for unit step input for the second-order system for ς/w_n=2 with different values of ς and w_n

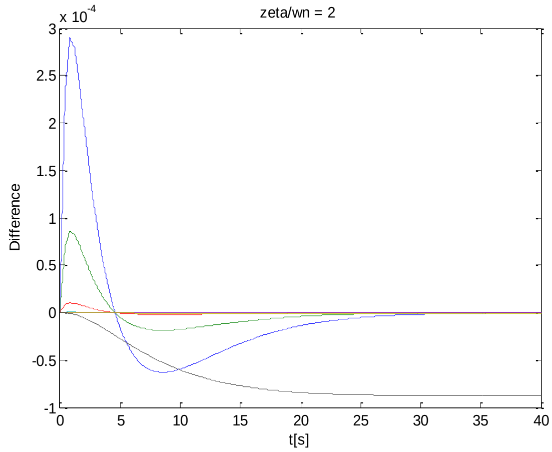

Fig. 20: Difference between output response for unit step input for the second-order system for ς/w_n=2 with different values of ς and w_n

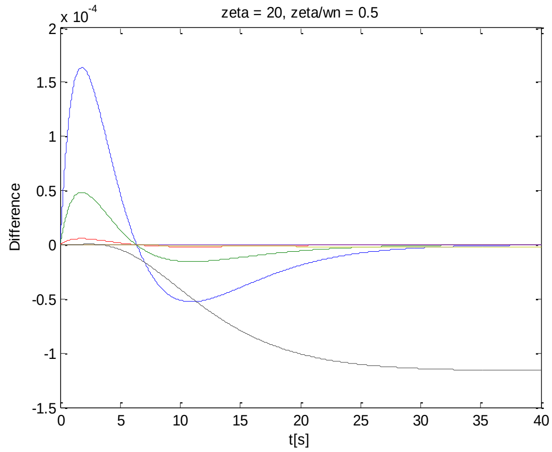

Fig. 21: Output response for unit step input for the second order system for ς/w_n= 0.5 with different values of ς and w_n.

To show this fact, the differences between the system responses are shown in Figure 20.

The difference between the output with constant ς/w_n is very small and does not exceed 0.0003.

Figure 21, shows the difference between the output of the second-order system when ς and w_n are varied with ς >20 and ς/w_n=0.5.From this figure, it is clear that the differences are very small and do not exceed 0.0003.

Based on Figure 20 and Figure 21, the output of the second-order system is the same ratio constant ς/w_n for a large value of ς, and for that reason, this ratio is the only required information to represent the system.

Figure 22 shows the response of the second-order system for ς =20 and w_n=5. From this figure t_d=5.5550s and t_s = 31.3250s.

Since the value of t_s/t_d = 5.638 , and from this value, it can be concluded that the value of ς is large, and then equ.(13) and equ.(14) can be used. From equ.(13), the value of ς/w_n =3.999, and this is the only information needed to represent the system. To verify the result, the response of the second-order system for ς =40, and w_n=10 to satisfy the ratio ς/w_n = 4. The difference between the output of the two systems is shown in Figure 23. Where the difference is very small.

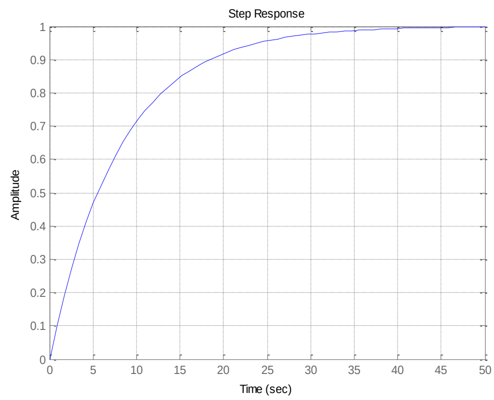

Fig. 22: Output response for unit step input for the second order system for ς =20 and w_n =5

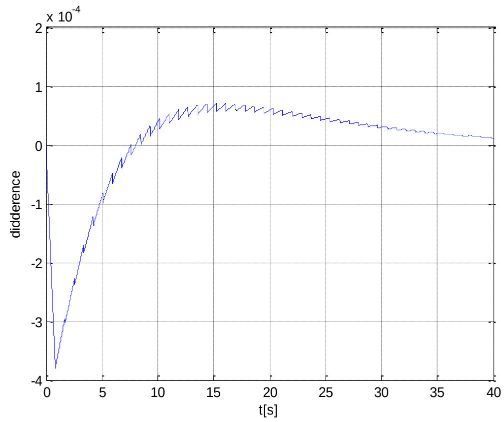

Fig. 23: Difference between output response for unit step input for the second order system (ς=20, w_n=5) and the proposed second order system (ς=40, w_n=10)

7 Conclusion

Mathematical equations for delay time and rise time, derived from numerical analysis of an overdamped second-order system, reveal a linear relationship with the damping factor, ς. Two distinct equations were formulated to address both low and high values of ς, and the results demonstrate the proposed model's accuracy, showing minimal modeling error. Using these equations, an equivalent second-order system can effectively model a higher-order overdamped system. For large ς values, it is shown that the system's response is primarily governed by the ratio ς/w_n, rather than by the individual values of ς and w_n. This ratio, which can be easily extracted from the model, is the key parameter needed for accurate system modeling.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this Paper, the author used Grammarly for language editing. After using this service, the author reviewed and edited the content as needed and take full responsibility for the content of the publication.

References:

[1] Haidekker, M. A. (2012). Solving Differential Equations in the Laplace Domain. *Linear Feedback Controls,* pp.27-56. doi: 10.1016/B978-0-12-405875-0.00003-6.

[2] Abdelaali, C., Bouklata, A., Benyassi, M., &Brouri, A. (2024). Identification of Linear Systems Having Time Delay Connected in Series. *WSEAS Transactions on Systems and Control*, vol.19, 234-239, http://dx.doi.org/10.37394/23203.2024.19.25.

[3] Sarwar, I. S. (2021). Modelling and Motion Control of BLDC motor for Pan tilt Platform. *WSEAS Transactions on Systems and Control,* vol.16, pp.183-193. http://dx.doi.org/10.37394/23203.2021.16.15.

[4] Shamseldin, M. A. (2021). Optimal COVID-19 based PD/PID cascaded tracking control for robot arm driven by BLDC motor. *WSEAS Transactions on Systems*, vol.20, pp.217-227. http://dx.doi.org/10.37394/23202.2021.20.24.

[5] Jakoubek, P, "Experimental identification of stabile non oscillatory systems from step responses by selected methods". *In Konference Studenske Tvuurci-Cinnosti, 2009,* Faculty of Mechanical Engineering, Czech Technical University in Prague.

[6] M. Schoukens and K. Tiels. Identification of block-oriented nonlinear systems starting from linear approximations: A survey, *Automatica,* Vol. 85, pp. 272-292, 2017. doi: 10.1016/j.automatica.2017.06.044.

[7] Boubaker, O., Balas, V. E., Benzaouia, A., Chaabane, M., Mahmoud, M. S., & Zhu, Q. (2017). Time-delay systems: Modeling, analysis, estimation, control, and synchronization. *Mathematical Problems in Engineering,* 2017(1), 1398904. https://doi.org/10.1155/2017/1398904.

[8] Maneechukate, T., K. Prompak, J. Koseeyaporn, and P. Wardkein. "Accuracy behavior explanation of electrical second-order system based on multi-time technique." *Scientific Research and Essays*, 8, no. 25 (2013): 1180-1190. doi: 10.5897/SRE2012.0755.

[9] Park, Ju H., Tae H. Lee, Yajuan Liu, and Jun Chen. Dynamic systems with time delays: *Stability and control.* Vol. 4. Singapore: Springer, 2019. doi: 10.1007/978-981-13-9254-2.

[10] Gontkovic, Daniel, and Robert Fonod. "Control and stability analyzing of the time-delay systems with time-varying delays." *Acta Electrotechnica et Informatica*, 11, no. 3 (2011): 70-74. doi: 10.2478/v10198-011-0032-9.

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Conflict of Interest

The authors have no conflicts of interest to declare.