Robust Controller Design for Large-scale Neutral Distributed-time-delay Systems

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Abstract: - Linear time-invariant neutral large-scale distributed-time-delay systems, composed of a number of overlapping subsystems, are considered. First, a robustness bound for such systems is derived. Then, a robust controller design approach, using this bound, is proposed. Excluding the derivation of this bound, the controller design is independent for each subsystem and is based on a local nominal model. Robust stability of the overall actual controlled system, however, is assured, once a scalar condition, which involves the derived bound, is satisfied.

Key-Words: - Controller design, Decentralized control, Large-scale systems, Time-delay systems, Neutral systems, Distributed time delay

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1 Introduction

There exist many examples of large-scale systems which involve time delays, [1]. Fundamentally, there exist two types of time delays: pointwise and distributed, [2]. Distributed time delay may be present in many application areas, such as neural networks, [3], [4], biology, [5], [6], traffic flow, [7], [8], logistics, [9], and combustion control, [10]. A time-delay system (whether a pointwise- or a distributed-time-delay system) may be *retarded* or *neutral*, [2]. A time-delay system is said to be neutral if it also involves time delays in the derivatives of its states, and is said to be retarded, otherwise. Thus, neutral systems are more general than retarded systems, [11].

There are many examples of large-scale systems, such as interconnected power systems, [12], data-communication networks, [13], freeway traffic regulation systems, [14], intelligent vehicle-highway systems, [15], and manufacturing systems, [16], which consist of subsystems with overlapping dynamics. The *overlapping decompositions* approach was first introduced in [17] for such systems. Although, initially, this approach was considered only for delay-free systems, since then both retarded and neutral systems with both pointwise and distributed time delay have also been considered, [18], [19].

In general, it is not possible to exactly model a practical system. Thus, any controller for a practical system should be *robust* against modeling uncertainties, [20]. Moreover, in the case of large-scale systems, it may be necessary or practical to ignore the interactions among subsystems during controller design, [21]. In this case, robustness against neglected interactions must also be considered. For this consideration, a measure was first introduced in [22]. This measure, which was called *robustness bound*, was then extended to also account for modelling uncertainties in [23]. However, [22] and [23] considered only delay-free systems. In the present work, we first derive such a robustness bound for linear time-invariant (LTI) neutral distributed-time-delay systems. Then, using this bound, we introduce a robust controller design approach for such systems. Excluding the derivation of this bound, the design approach is independent for each subsystem and is based on a local nominal model. However, satisfying a scalar condition, involving the derived bound, assures robust stability of the overall actual controlled system.

Throughout, $\mathbf{R}$ and $\mathbf{C}$ respectively denote the sets of real and complex numbers. For $s \in \mathbf{C}$, $\mathrm{Re}(s)$

denotes the real part of s. For $\mu \in \mathbf{R}$, $\mathbf{C}_\mu^+ := \{s \in \mathbf{C} \mid \mathrm{Re}(s) \geq \mu\}$. For positive integers k and l, $\mathbf{R}^k$ and $\mathbf{R}^{k\times l}$ denote the spaces of, respectively, k-dimensional real vectors and $k \times l$-dimensional real matrices. I_k denotes the $k \times k$-dimensional identity matrix and I denotes the identity matrix of appropriate dimensions. 0 may denote either the scalar zero, a zero matrix, or a matrix function which is identically zero. For a positive integer N, $\bar{N} := \{1, 2, \ldots, N\}$. The imaginary unit, $\sqrt{-1}$, is denoted by j. The determinant, the maximum singular value, and the minimum singular value of a matrix are respectively denoted by $\det(\cdot)$, $\bar{\sigma}(\cdot)$, and $\underline{\sigma}(\cdot)$. For $M(\cdot) : [-\theta, 0] \to \mathbf{R}^{k\times l}$, $\|M\| := \int_{-\theta}^{0} \bar{\sigma}\left(M(\phi)\right) d\phi$. Number of elements of a set S is denoted by $\#(S)$. Finally, a block diagonal matrix is denoted by $\mathrm{bdiag}(\cdots)$.

2 Problem Statement

In this work, as mentioned in the introduction, we consider large-scale LTI neutral distributed-time-delay systems, which are composed of a number of overlapping subsystems. In practice, there may be very different structures of overlapping. For brevity however, here we only consider systems with a number (N) of subsystems interconnected through an *overlapping part*. Such a system can compactly be described as:

$$\dot{x}(t) + \int_{-\theta}^{0} E(\phi)\dot{x}(t+\phi)d\phi = \int_{-\theta}^{0} \left(A(\phi)x(t+\phi) + B(\phi)u(t+\phi)\right) d\phi \quad (1)$$

$$y(t) = \int_{-\theta}^{0} C(\phi)x(t+\phi)d\phi \quad (2)$$

where t is the time variable and $\theta > 0$ is the maximum time-delay. The state, the input, and the output vectors are partitioned as:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ x_c \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad (3)$$

where $x_i \in \mathbf{R}^{n_i}$ is the state, $u_i \in \mathbf{R}^{m_i}$ is the input, and $y_i \in \mathbf{R}^{r_i}$ is the output vector of the i^{th} subsystem ($i \in \bar{N}$). Furthermore, $x_c \in \mathbf{R}^{n_c}$ is the state vector for the overlapping dynamics. The matrix functions $E(\cdot) : [-\theta, 0] \to \mathbf{R}^{n\times n}$, $A(\cdot) : [-\theta, 0] \to \mathbf{R}^{n\times n}$, $B(\cdot) : [-\theta, 0] \to \mathbf{R}^{n\times m}$, and $C(\cdot) : [-\theta, 0] \to \mathbf{R}^{r\times n}$ (where $n := \sum_{i=1}^{N} n_i + n_c$, $m := \sum_{i=1}^{N} m_i$, and $r := \sum_{i=1}^{N} r_i$) are also decomposed as:

$$E(\cdot) = \begin{bmatrix} E_1(\cdot) & & 0 & E_{1c}(\cdot) \\ & \ddots & & \vdots \\ 0 & & E_N(\cdot) & E_{Nc}(\cdot) \\ E_{c1}(\cdot) & \cdots & E_{cN}(\cdot) & E_c(\cdot) \end{bmatrix},$$

$$A(\cdot) = \begin{bmatrix} A_1(\cdot) & & 0 & A_{1c}(\cdot) \\ & \ddots & & \vdots \\ 0 & & A_N(\cdot) & A_{Nc}(\cdot) \\ A_{c1}(\cdot) & \cdots & A_{cN}(\cdot) & A_c(\cdot) \end{bmatrix},$$

$$B(\cdot) = \begin{bmatrix} B_1(\cdot) & & 0 \\ & \ddots & \\ 0 & & B_N(\cdot) \\ 0 & \cdots & 0 \end{bmatrix},$$

and

$$C(\cdot) = \begin{bmatrix} C_1(\cdot) & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & C_N(\cdot) & 0 \end{bmatrix},$$

where the partitionings are compatible with those in (3). All the submatrix functions shown above are assumed to be bounded. Furthermore, the above submatrix functions, except for the output submatrix functions (i.e., submatrix functions of $C(\cdot)$), are assumed to be uncertain. Thus, representing any one of these submatrix functions by $M_\bullet(\cdot)$, we assume that $M_\bullet(\cdot) = M_\bullet^n(\cdot) + M_\bullet^u(\cdot)$, where $M_\bullet^n(\cdot)$ is the nominal part and $M_\bullet^u(\cdot)$ is the uncertain part. The uncertain parts, however, are assumed to be bounded as follows:

$$\begin{array}{lll} \|E_i^u\| \leq \epsilon_i\,, & \|A_i^u\| \leq \alpha_i\,, & \|B_i^u\| \leq \beta_i\,, \\ \|E_{ic}^u\| \leq \epsilon_{ic}\,, & \|E_{ci}^u\| \leq \epsilon_{ci}\,, & \|E_c^u\| \leq \epsilon_c\,, \\ \|A_{ic}^u\| \leq \alpha_{ic}\,, & \|A_{ci}^u\| \leq \alpha_{ci}\,, & \|A_c^u\| \leq \alpha_c\,, \end{array} \quad (4)$$

where ϵ_c, α_c, and ϵ_i, ϵ_{ic}, ϵ_{ci}, α_i, α_{ic}, α_{ci}, and β_i, $i \in \bar{N}$, are known non-negative bounds. Here, without loss of generality, we assume that all the input-output uncertainties are shown at the input, hence there are no uncertainties in $C(\cdot)$.

Before going any further, for the sake of completeness, we note that a system of the form (1) is *stable* if and only if it does not have any modes in $\mathbf{C}_{-\mu}^+$ for some $\mu > 0$, [24]. $s_o \in \mathbf{C}$ is said to be a *mode* of (1) if $\det\left(s_o\bar{E}(s_o) - \bar{A}(s_o)\right) = 0$, where

$$\bar{E}(s) := I + \int_{-\theta}^{0} E(\phi)e^{s\phi}d\phi$$

and

$$\bar{A}(s) := \int_{-\theta}^{0} A(\phi) e^{s\phi} d\phi \ .$$

A mode s_o is called as an *unstable mode* if $s_o \in \mathbf{C}_0^+$. Although (1) may have infinitely many modes, for any $\mu > 0$, it has only finitely many modes in $\mathbf{C}_{\psi+\mu}^+$, [24], where

$$\psi := \sup \left\{ \mathrm{Re}(s) \mid \det \left(\bar{E}(s) \right) = 0 \right\} \ . \qquad (5)$$

Therefore, (1) has only finitely many unstable modes if $\psi < 0$. Since it is not possible to stabilize a system of the form (1)–(2) with $\psi \geq 0$, by using a proper LTI controller, [25], we will assume that system (1) satisfies $\psi < 0$.

We aim to design decentralized (possibly time-delay) LTI controlers with transfer function matrices (TFMs) $K_i(s)$ to be applied in a negative feedback configuration from $y_i(t)$ to $u_i(t)$ ($i \in \bar{N}$), such that:

(i) $K_i(s)$ stabilizes the i^{th} local nominal model,
(ii) $K_i(s)$ satisfies a local performance requirement for the i^{th} local nominal model, and
(iii) Robust stability of the actual overall controlled system is assured for all uncertainties that satisfy (4).

Furthermore, design of each $K_i(s)$ should be based on a *local nominal model*.

3 Local Nominal Models

In order to obtain a local nominal model, we *expand*, [19], the above system by using the transformation

$$T = \begin{bmatrix} I_{n_1} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & I_{n_c} \\ 0 & I_{n_2} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & I_{n_c} \\ \vdots & & & & & & \\ 0 & 0 & \cdots & \cdots & 0 & I_{n_N} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & I_{n_c} \end{bmatrix} .$$

The *expanded system* is described as:

$$\dot{\hat{x}}(t) + \int_{-\theta}^{0} \hat{E}(\phi) \dot{\hat{x}}(t+\phi) d\phi = \int_{-\theta}^{0} \left(\hat{A}(\phi) \hat{x}(t+\phi) + \hat{B}(\phi) \hat{u}(t+\phi) \right) d\phi \qquad (6)$$

$$\hat{y}(t) = \int_{-\theta}^{0} \hat{C}(\phi) \hat{x}(t+\phi) d\phi \qquad (7)$$

where

$$\hat{E}(\phi) := \begin{bmatrix} \hat{E}_1(\phi) & \hat{E}_{12}(\phi) & \cdots & \hat{E}_{1N}(\phi) \\ \hat{E}_{21}(\phi) & \hat{E}_2(\phi) & \cdots & \hat{E}_{2N}(\phi) \\ \vdots & & & \\ \hat{E}_{N1}(\phi) & \hat{E}_{N2}(\phi) & \cdots & \hat{E}_N(\phi) \end{bmatrix} ,$$

$$\hat{A}(\phi) := \begin{bmatrix} \hat{A}_1(\phi) & \hat{A}_{12}(\phi) & \cdots & \hat{A}_{1N}(\phi) \\ \hat{A}_{21}(\phi) & \hat{A}_2(\phi) & \cdots & \hat{A}_{2N}(\phi) \\ \vdots & & & \\ \hat{A}_{N1}(\phi) & \hat{A}_{N2}(\phi) & \cdots & \hat{A}_N(\phi) \end{bmatrix} ,$$

$$\hat{B}(\phi) := \mathrm{bdiag} \left(\hat{B}_1(\phi), \ldots, \hat{B}_N(\phi) \right) ,$$

and

$$\hat{C}(\phi) := \mathrm{bdiag} \left(\hat{C}_1(\phi), \ldots, \hat{C}_N(\phi) \right) ,$$

where, for $i \in \bar{N}$ and $k \in \bar{N} \setminus \{i\}$,

$$\hat{E}_i(\phi) := \begin{bmatrix} E_i(\phi) & E_{ic}(\phi) \\ E_{ci}(\phi) & E_c(\phi) \end{bmatrix} ,$$

$$\hat{A}_i(\phi) := \begin{bmatrix} A_i(\phi) & A_{ic}(\phi) \\ A_{ci}(\phi) & A_c(\phi) \end{bmatrix} ,$$

$$\hat{E}_{ik}(\phi) := \begin{bmatrix} 0 & 0 \\ E_{ck}(\phi) & 0 \end{bmatrix} ,$$

$$\hat{A}_{ik}(\phi) := \begin{bmatrix} 0 & 0 \\ A_{ck}(\phi) & 0 \end{bmatrix} ,$$

$$\hat{B}_i(\phi) := \begin{bmatrix} B_i(\phi) \\ 0 \end{bmatrix} ,$$

and

$$\hat{C}_i(\phi) := \begin{bmatrix} C_i(\phi) & 0 \end{bmatrix} .$$

Note that relations $\hat{E}(\phi)T = TE(\phi)$, $\hat{A}(\phi)T = TA(\phi)$, $\hat{B}(\phi) = TB(\phi)$, and $\hat{C}(\phi)T = C(\phi)$, for all $\phi \in [-\theta, 0]$, are satisfied. This implies that the given system (1)–(2) is *included by* the *expanded system* (6)–(7), [19]. This, in turn, implies that the input-output map of both systems is the same, [19].

We note that the expanded system (6)–(7) is composed of N disjoint subsystems with only *weak* interactions (through $\hat{E}_{ik}(\cdot)$ and $\hat{A}_{ik}(\cdot)$). Thus, a *local model* can be obtained by ignoring these interactions as:

$$\dot{\hat{x}}_i(t) + \int_{-\theta}^{0} \hat{E}_i(\phi) \dot{\hat{x}}_i(t+\phi) d\phi = \int_{-\theta}^{0} \left(\hat{A}_i(\phi) \hat{x}_i(t+\phi) + \hat{B}_i(\phi) \hat{u}_i(t+\phi) \right) d\phi \qquad (8)$$

$$\hat{y}_i(t) = \int_{-\theta}^{0} \hat{C}_i(\phi) \hat{x}_i(t+\phi) d\phi \qquad (9)$$

A *local nominal model* can then be obtained by taking the nominal part of (8)–(9):

$$\dot{\hat{x}}_i(t) + \int_{-\theta}^{0} \hat{E}_i^n(\phi)\dot{\hat{x}}_i(t+\phi)d\phi = \int_{-\theta}^{0} \Big(\hat{A}_i^n(\phi)\hat{x}_i(t+\phi) + \hat{B}_i^n(\phi)\hat{u}_i(t+\phi)\Big)\, d\phi \quad (10)$$

$$\hat{y}_i(t) = \int_{-\theta}^{0} \hat{C}_i(\phi)\hat{x}_i(t+\phi)d\phi \quad (11)$$

where

$$\hat{E}_i^n(\phi) := \begin{bmatrix} E_i^n(\phi) & E_{ic}^n(\phi) \\ E_{ci}^n(\phi) & E_c^n(\phi) \end{bmatrix} ,$$

$$\hat{A}_i^n(\phi) := \begin{bmatrix} A_i^n(\phi) & A_{ic}^n(\phi) \\ A_{ci}^n(\phi) & A_c^n(\phi) \end{bmatrix} ,$$

and

$$\hat{B}_i^n(\phi) := \begin{bmatrix} B_i^n(\phi) \\ 0 \end{bmatrix} .$$

Here, it is assumed that

$$\hat{\psi} := \sup\Big\{\mathrm{Re}(s) \mid \det\Big(\bar{\hat{E}}(s)\Big) = 0\Big\} < 0 , \quad (12)$$

where

$$\bar{\hat{E}}(s) := I + \int_{-\theta}^{0} \hat{E}(\phi)e^{s\phi}d\phi , \quad (13)$$

and

$$\hat{\psi}_i^n := \sup\Big\{\mathrm{Re}(s) \mid \det\Big(\bar{\hat{E}}_i^n(s)\Big) = 0\Big\} < 0 , \quad (14)$$

$\forall i \in \bar{N}$, where

$$\bar{\hat{E}}_i^n(s) := I + \int_{-\theta}^{0} \hat{E}_i^n(\phi)e^{s\phi}d\phi . \quad (15)$$

It is further assumed that

$$\#(\hat{S}) = \sum_{i=1}^{N} \#(S_i) , \quad (16)$$

where $\hat{S}$ denotes the set of unstable modes of the expanded system (6)–(7) and S_i denotes the set of unstable modes of the i^{th} local nominal model (10)–(11). We note that both $\#(\hat{S})$ and $\#(S_i)$ are finite by assumptions (12) and (14), respectively.

4 Robustness Bound

A robustness bound is derived in this section. This bound is to account for the

- uncertainties in the subsystem models,
- neglected interactions betwen the subsystem models, and
- uncertainties in the interactions between the subsystem models.

For this, from (10)–(11), the TFM of the i^{th} local nominal model is obtained as

$$\Gamma_i^n(s) = \bar{\hat{C}}_i(s)\Big(s\bar{\hat{E}}_i^n(s) - \bar{\hat{A}}_i^n(s)\Big)^{-1}\bar{\hat{B}}_i^n(s) \quad (17)$$

where $\bar{\hat{C}}_i(s) := \int_{-\theta}^{0}\hat{C}_i(\phi)e^{s\phi}d\phi$, $\bar{\hat{A}}_i^n(s) := \int_{-\theta}^{0}\hat{A}_i^n(\phi)e^{s\phi}d\phi$, $\bar{\hat{B}}_i^n(s) := \int_{-\theta}^{0}\hat{B}_i^n(\phi)e^{s\phi}d\phi$, and $\bar{\hat{E}}_i^n(s)$ is defined in (15).

The TFM for the *overall design model* is then given as

$$\Gamma^n(s) = \mathrm{bdiag}\left(\Gamma_1^n(s), \ldots, \Gamma_N^n(s)\right) . \quad (18)$$

On the other hand, since the system (1)–(2) is included by the system (6)–(7), the TFM, $\Gamma(s)$, of the former system is same as the TFM, $\hat{\Gamma}(s)$, of the latter system:

$$\Gamma(s) = \hat{\Gamma}(s) = \bar{\hat{C}}(s)\Big(s\bar{\hat{E}}(s) - \bar{\hat{A}}(s)\Big)^{-1}\bar{\hat{B}}(s) \quad (19)$$

where $\bar{\hat{A}}(s) := \int_{-\theta}^{0}\hat{A}(\phi)e^{s\phi}d\phi$, $\bar{\hat{B}}(s) := \int_{-\theta}^{0}\hat{B}(\phi)e^{s\phi}d\phi$, $\bar{\hat{C}}(s) := \int_{-\theta}^{0}\hat{C}(\phi)e^{s\phi}d\phi = \mathrm{bdiag}\Big(\bar{\hat{C}}_1(s), \ldots, \bar{\hat{C}}_N(s)\Big)$, and $\bar{\hat{E}}(s)$ is defined in (13).

Now, let $\Lambda(s)$ be such that:

$$\Gamma(s) = \Gamma^n(s)\left(I + \Lambda(s)\right) . \quad (20)$$

As it was first shown in [23], for delay-free systems, an upper bound of the norm of $\Lambda(j\omega)$ defines a robustness bound. Such a bound can be obtained as follows:

Lemma 1: Suppose that $\lambda_d(\omega) > 0$, $\forall\omega \in \mathbf{R}$. Then

$$\bar{\sigma}(\Lambda(j\omega)) \le \lambda(\omega) := \frac{\lambda_n(\omega)}{\lambda_d(\omega)} , \quad \forall\omega \in \mathbf{R} , \quad (21)$$

where

$$\lambda_d(\omega) := \underline{\sigma}\left(L(j\omega)\right) - \sum_{k=0}^{N-1}\max_{l\in\bar{N}}\Big\{\lambda_d^{l,[l+k]}(\omega)\Big\} , \quad (22)$$

where

$$L(s) := \begin{bmatrix} L_{11}(s) & L_{12}(s) & \cdots & L_{1N}(s) \\ L_{21}(s) & L_{22}(s) & \cdots & L_{2N}(s) \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1}(s) & L_{N2}(s) & \cdots & L_{NN}(s) \end{bmatrix} , \tag{23}$$

where, for $l \in \bar{N}$ and $k \in \bar{N} \setminus l$,

$$L_{ll}(s) := \bar{\hat{B}}_l^n(s) ,$$

$$L_{lk}(s) := Q_{lk}(s) G_k(s) ,$$

where

$$Q_{lk}(s) := \begin{bmatrix} 0 & 0 \\ \int_{-\theta}^{0} \left(s E_{ck}^n(\phi) - A_{ck}^n(\phi) \right) e^{s\phi} d\phi & 0 \end{bmatrix}$$

and

$$G_l(s) := \left(s\bar{\hat{E}}_l^n(s) - \bar{\hat{A}}_l^n(s) \right)^{-1} \bar{\hat{B}}_l^n(s) ,$$

$$[k] := \begin{cases} k , & \text{if } k \le N \\ k - N , & \text{otherwise} \end{cases} ,$$

$$\lambda_d^{l,l}(\omega) := \left(|\omega| \hat{\epsilon}_l + \hat{\alpha}_l \right) g_l(\omega) ,$$

where

$$g_l(\omega) := \bar{\sigma} \left(G_l(j\omega) \right) ,$$

$$\hat{\epsilon}_l := \max(\epsilon_l, \epsilon_c) + \max(\epsilon_{cl}, \epsilon_{lc}) ,$$

and

$$\hat{\alpha}_l := \max(\alpha_l, \alpha_c) + \max(\alpha_{cl}, \alpha_{lc}) ,$$

and

$$\lambda_d^{l,k}(\omega) := \left(|\omega| \epsilon_{ck} + \alpha_{ck} \right) g_k(\omega) ,$$

and

$$\lambda_n(\omega) := \sum_{k=0}^{N-1} \max_{l \in \bar{N}} \left\{ \lambda_n^{l,[l+k]}(\omega) \right\} , \tag{24}$$

where, for $l \in \bar{N}$ and $k \in \bar{N} \setminus l$,

$$\lambda_n^{l,l}(\omega) := \beta_l + \left(|\omega| \hat{\epsilon}_l + \hat{\alpha}_l \right) g_l(\omega)$$

and

$$\lambda_n^{l,k}(\omega) := \left(\bar{\sigma} \left(Q_{lk}(j\omega) \right) + |\omega| \epsilon_{ck} + \alpha_{ck} \right) g_k(\omega) .$$

Proof: From (20), we can let $\Lambda(s)$ satisfy

$$\left(s\bar{\hat{E}}(s) - \bar{\hat{A}}(s) \right)^{-1} \bar{\hat{B}}(s)$$
$$= \left(s\bar{\hat{E}}^n(s) - \bar{\hat{A}}^n(s) \right)^{-1} \bar{\hat{B}}^n(s) \left(I + \Lambda(s) \right) \tag{25}$$

where

$$\bar{\hat{E}}^n(s) := \text{bdiag} \left(\bar{\hat{E}}_1^n(s), \ldots, \bar{\hat{E}}_N^n(s) \right) ,$$

$$\bar{\hat{A}}^n(s) := \text{bdiag} \left(\bar{\hat{A}}_1^n(s), \ldots, \bar{\hat{A}}_N^n(s) \right) ,$$

and

$$\bar{\hat{B}}^n(s) := \text{bdiag} \left(\bar{\hat{B}}_1^n(s), \ldots, \bar{\hat{B}}_N^n(s) \right) .$$

Now, premultiply both sides of (25) by $\left(s\bar{\hat{E}}(s) - \bar{\hat{A}}(s) \right)$ and rearange terms to obtain $N(s) = D(s)\Lambda(s)$, where

$$D(s) := \left(s\bar{\hat{E}}(s) - \bar{\hat{A}}(s) \right) G(s)$$

and

$$N(s) := \bar{\hat{B}}(s) - D(s) ,$$

where $G(s) := \text{bdiag} \left(G_1(s), \ldots, G_N(s) \right)$. Then, note that $\bar{\sigma}(N(j\omega)) \le \lambda_n(\omega)$ and $\underline{\sigma}(D(j\omega)) \ge \lambda_d(\omega)$. These, however, imply (21). □

Remark: (21) is not defined if $\lambda_d(\omega) \le 0$, that is if $\underline{\sigma}(L(j\omega)) \le \sum_{k=0}^{N-1} \max_{l \in \bar{N}} \left\{ \lambda_d^{l,[l+k]}(\omega) \right\}$, for some $\omega \in \mathbf{R}$. However, $L(j\omega)$ contains the nominal input terms $\bar{\hat{B}}_l^n(j\omega)$ on the diagonal blocks and interaction terms $Q_{lk}(j\omega)$ on the off-diagonal blocks. Since the norm of $\bar{\hat{B}}_l^n(j\omega)$, for any $\omega \in \mathbf{R}$, is typically much larger than the norm of the interaction terms and the terms $\lambda_d^{l,k}(\omega)$, which contain bounds on the uncertainties, are typically small, typically,

$$\underline{\sigma}(L(j\omega)) > \sum_{k=0}^{N-1} \max_{l \in \bar{N}} \left\{ \lambda_d^{l,[l+k]}(\omega) \right\} ,$$

and hence $\lambda_d(\omega) > 0$.

5 Controller Design

In this section we consider the problem of designing decentralized controlers $K_i(s)$, $i \in \bar{N}$, so that the requirements (i)–(iii) given at the end of Section 2 are satisfied.

Note that requirements (i) and (ii) are local requirements. Thus, each controller $K_i(s)$ can be designed independently to satisfy these two requirements. To satisfy requirement (iii), the i^{th} controller, $K_i(s)$, can be designed to satisfy

$$\bar{\sigma} \left(T_i^n(j\omega) \right) < \frac{1}{\lambda(\omega)} , \quad \forall \omega \in \mathbf{R} , \ \forall i \in \bar{N} , \tag{26}$$

where

$$T_i^n(s) := \Gamma_i^n(s)K_i(s)\left[I + \Gamma_i^n(s)K_i(s)\right]^{-1} \quad (27)$$

and $\lambda(\omega)$ is given by (21). The following theorem shows that when requirement (i) is satisfied for each $i \in \bar{N}$, satisfying (26) assures robust stability of the actual overall controlled system.

Theorem 1: Suppose that (12), (14), and (16) hold. Also suppose that $K_i(s)$ stabilizes the i^{th} local nominal model (10)–(11), for each $i \in \bar{N}$, and that (26) holds. Then, the actual overall system under the designed decentralized controllers is robustly stable for all uncertainties that satisfy (4).

Proof: The block diagonal structure (18) of the overall design model, together with (14), implies that there are no modes of the overall design model approaching the imaginary axis from left and that $\#(\hat{S}^n)$ is finite, where $\hat{S}^n$ is the set of unstable modes of the overall design model. Assumption (12), on the other hand, implies that there are no modes of the expanded open-loop system approaching the imaginary axis from left and that $\#(\hat{S})$ is finite. For the overall closed-loop design model, the complementary sensitivity matrix is

$$\begin{aligned} T^n(s) &= \Gamma^n(s)K(s)\left[I + \Gamma^n(s)K(s)\right]^{-1} \\ &= \text{bdiag}\left(T_1^n(s), \ldots, T_N^n(s)\right) \end{aligned} \quad (28)$$

where $K(s) := \text{bdiag}\left(K_1(s), \ldots, K_N(s)\right)$ is the TFM for the overall controller and $T_i^n(s)$, $i \in \bar{N}$, is given by (27). Because of the block diagonal structure of the overall closed-loop design model, requirement (i) implies stability of the overall design model under the designed controllers. The block diagonal structure of the overall design model, on the other hand, implies that

$$\#(\hat{S}^n) = \sum_{i=1}^{N} \#(S_i) \; . \quad (29)$$

Then, by (16),

$$\#(\hat{S}) = \#(\hat{S}^n) \; . \quad (30)$$

Furthermore, by (28),

$$\bar{\sigma}\left(T^n(j\omega)\right) = \max_{i \in \bar{N}} \left\{\bar{\sigma}\left(T_i^n(j\omega)\right)\right\} \; .$$

Thus, when (26) holds, by (21), we have

$$\bar{\sigma}\left(T^n(j\omega)\right) < \frac{1}{\bar{\sigma}\left(\Lambda(j\omega)\right)} \; , \quad \forall \omega \in \mathbf{R} \; . \quad (31)$$

This, together with (30), however, implies that the expanded system (6)–(7) under the designed controllers is robustly stable, [20]. However, since the system (1)–(2) is included by the system (6)–(7), this implies the robust stability of the actual system (1)–(2) under the designed controllers, [19]. □

Note that, once the robustness bound (21) is obtained, (26) is in fact a local requirement. This means that each local controller $K_i(s)$ can be designed independently of others based on the local nominal model (10)–(11). For this, any centralized controller design approach developed for LTI time-delay systems (e.g., [10], [26], [27], [28], [29]) can be used.

6 Example

We consider a system with $N = 2$, $n_1 = n_2 = n_c = m_1 = m_2 = r_1 = r_2 = 1$. The time is assumed to be normalized so that the maximum time delay is $\theta = 1$. The system is decribed by (1)–(2) with $E_1^n(\phi) = E_2^n(\phi) = e^{\phi}$, $E_c^n(\phi) = E_{1c}^n(\phi) = E_{2c}^n(\phi) = 0$, $E_{c1}^n(\phi) = E_{c2}^n(\phi) = \frac{1}{2}e^{\phi}$, $A_1^n(\phi) = A_2^n(\phi) = 2e^{\phi}$, $A_c^n(\phi) = -2e^{\phi}$, $A_{1c}^n(\phi) = A_{2c}^n(\phi) = A_{c1}^n(\phi) = A_{c2}^n(\phi) = \frac{1}{4}e^{\phi}$, $B_1^n(\phi) = B_2^n(\phi) = 2 - \phi$, and $C_1(\phi) = C_2(\phi) = -\phi$, for $\phi \in [-1, 0]$. The bounds in (4) are assumed to be $\epsilon_1 = \epsilon_2 = \epsilon_c = \epsilon_{1c} = \epsilon_{2c} = \epsilon_{c1} = \epsilon_{c2} = \alpha_1 = \alpha_2 = \alpha_c = \alpha_{1c} = \alpha_{2c} = \alpha_{c1} = \alpha_{c2} = \beta_1 = \beta_2 = 0.01$. The local nominal models are obtained as

$$\begin{aligned} &\dot{\hat{x}}_i(t) + \int_{-1}^{0} \begin{bmatrix} e^{\phi} & 0 \\ \frac{1}{2}e^{\phi} & 0 \end{bmatrix} \dot{\hat{x}}_i(t+\phi)d\phi \\ &\quad = \int_{-1}^{0} \left(\begin{bmatrix} 2e^{\phi} & \frac{1}{4}e^{\phi} \\ \frac{1}{4}e^{\phi} & -2e^{\phi} \end{bmatrix} \hat{x}_i(t+\phi) \right. \\ &\qquad \left. + \begin{bmatrix} 2-\phi \\ 0 \end{bmatrix} \hat{u}_i(t+\phi) \right) d\phi \end{aligned} \quad (32)$$

$$\hat{y}_i(t) = \int_{-1}^{0} \begin{bmatrix} -\phi & 0 \end{bmatrix} \hat{x}_i(t+\phi)d\phi \quad (33)$$

which are identical for $i = 1$ and 2.

First, using (14), we calculate $\hat{\psi}_i^n = -2.5321$, which means that (14) holds. We then find that $\hat{\psi} \leq -2.4987$ for all uncertainties in $\hat{E}(\phi)$ which satisfy (4). Thus, (12) also holds. Next, we find that (32) has exactly one unstable mode at $s = 0.6547$. We also find that the only unstable modes of the expanded system (6), with all the uncertainties set to zero, are $s = 0.6714 \pm j0.1655$. We also verify that

$$\det\left(j\omega\bar{\hat{E}}(j\omega) - \bar{\hat{A}}(j\omega)\right) \neq 0 \; , \quad \forall \omega \in \mathbf{R} \; ,$$

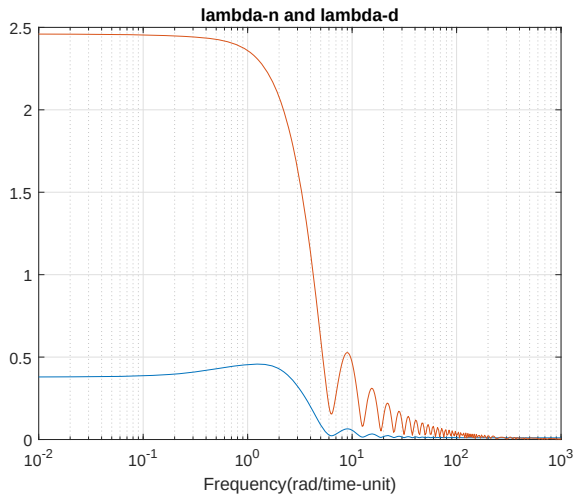

Fig. 1. $\lambda_n(\omega)$ (in blue - plot with lower low-frequency gain) and $\lambda_d(\omega)$ (in red - plot with higher low-frequency gain).

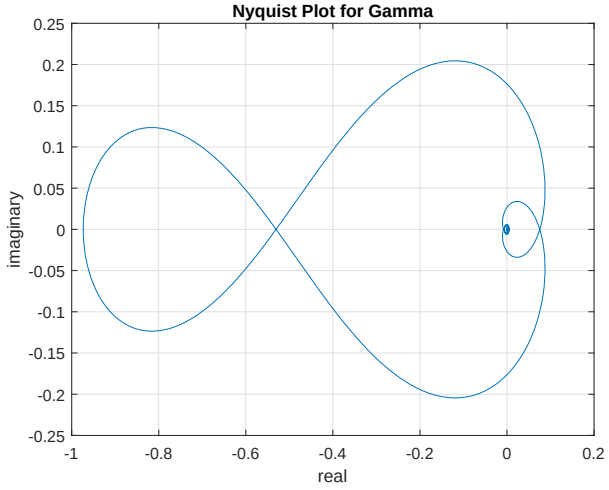

Fig. 2. Nyquist plot of $\Gamma_1^n(j\omega) = \Gamma_2^n(j\omega)$.

for all uncertainties in $\hat{E}(\phi)$ and $\hat{A}(\phi)$ which satisfy (4). This implies that the expanded system (6) has exactly two unstable modes for all uncertainties that satisfy (4). Thus, (16) also holds.

Next, we calculate $\lambda_n(\omega)$ and $\lambda_d(\omega)$, using (24) and (22), respectively, and plot them in Fig. 1. We note that $\lambda_d(\omega) > 0$, $\forall \omega \in \mathbf{R}$. Then, in order to design controllers to stabilize each local nominal model (32)–(33), we draw the Nyquist plot of $\Gamma_1^n(j\omega) = \Gamma_2^n(j\omega)$ in Fig. 2. Note that the Nyquist plot of $k\Gamma_i^n(j\omega)$ $(i = 1, 2)$ would encircle the -1 point once in the counter-clockwise direction if k is chosen in the range:

$$1.0272 = \frac{1}{0.9735} < k < \frac{1}{0.5305} = 1.8850 \ . \quad (34)$$

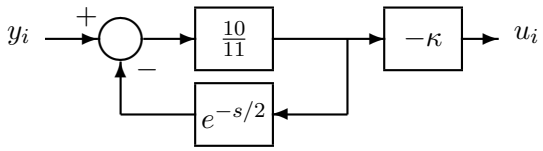

Fig. 3. Implementation of the controller $K_i(s)$.

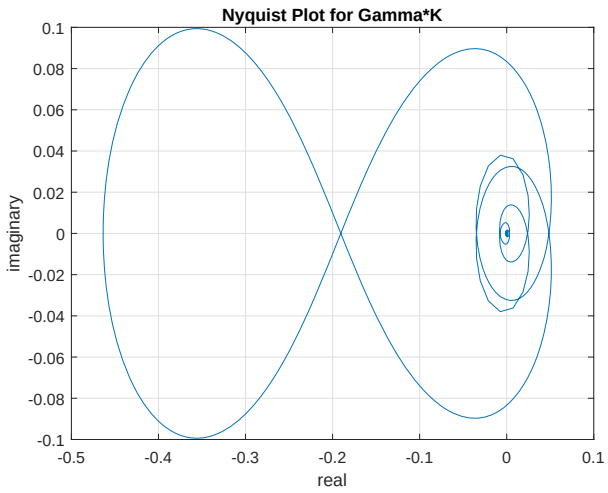

Fig. 4. Nyquist plot of $\Gamma_i^n(j\omega)K_i(j\omega)$ for $\kappa = 1$.

Recalling that (32) has exactly one unstable mode, the Nyquist theorem, [30], implies that static controllers

$$K_1(s) = K_2(s) = k \ , \quad (35)$$

with k chosen in the range (34), stabilize the local nominal models (32)–(33). However, (26) is not satisfied for any k in the range (34). This implies that, although static controllers can stabilize the local nominal models, they can not guarantee the robust stability of the overall system.

Next, we try time-delay controllers of the form

$$K_1(s) = K_2(s) = \frac{\kappa}{e^{-s/2} + 1.1} \ , \quad (36)$$

which can be implemented using a delay element of delay value equal to the half of the maximum delay in the system, as shown in Fig. 3. Note that, (36) defines a stable controller, since $e^{-s/2} + 1.1 \neq 0$, $\forall s \in \mathbf{C}_0^+$. Nyquist plot of $\Gamma_i^n(j\omega)K_i(j\omega)$ $(i = 1, 2)$ for $\kappa = 1$ is shown in Fig. 4. This plot indicates that the plot of $\Gamma_i^n(j\omega)K_i(j\omega)$ would encircle the -1 point once in the counter-clockwise direction if κ is chosen in the range:

$$2.1517 = \frac{1}{0.4636} < \kappa < \frac{1}{0.1906} = 5.2470 \ . \quad (37)$$

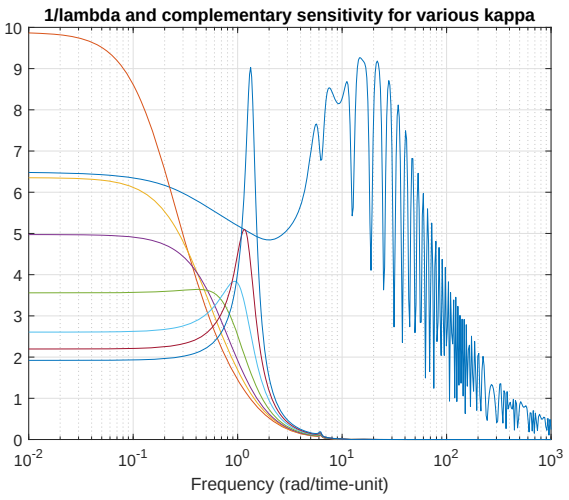

Fig. 5. Plots of $1/\lambda(\omega)$ (in blue - plot with second highest low-frequency gain) and of $\bar{\sigma}\left(T_1^n(j\omega)\right) = \bar{\sigma}\left(T_2^n(j\omega)\right)$ for $\kappa = 2.4$ (in red - plot with highest low-frequency gain), $\kappa = 2.56$ (in yellow - plot with third highest low-frequency gain), $\kappa = 2.7$ (in purple - plot with fourth highest low-frequency gain), $\kappa = 3$ (in green - plot with fourth lowest low-frequency gain), $\kappa = 3.5$ (in light blue - plot with third lowest low-frequency gain), $\kappa = 3.96$ (in brown - plot with second lowest low-frequency gain), and $\kappa = 4.5$ (in cyan - plot with lowest low-frequency gain).

Therefore, controllers (36), with κ chosen in the range (37), stabilize the local nominal models (32)–(33). Next, we plot $\bar{\sigma}\left(T_1^n(j\omega)\right) = \bar{\sigma}\left(T_2^n(j\omega)\right)$, where $T_i^n(s)$ is given by (27), with $K_i(s)$ as in (36), for various κ in the range (37), together with $\frac{1}{\lambda(\omega)} = \frac{\lambda_d(\omega)}{\lambda_n(\omega)}$, in Fig. 5. This figure indicates that (26) is satisfied with κ chosen in the range:

$$2.56 < \kappa < 3.96 \; . \tag{38}$$

Therefore, by Theorem 1, decentralized controllers (36), with κ chosen in the range (38), robustly stabilize the actual overall system for all uncertainties that satisfy (4). In particular, for $\kappa = 3$ (see the related curve in Fig. 5), a wide robustness margin for the overall system and a smooth response for (32)–(33) would be obtained. Thus, all the requirements (i)–(iii) would be satisfied.

7 Conclusions

Robust controller design has been considered for large-scale LTI neutral distributed-time-delay systems, composed of a number of overlapping subsystems. A frequency dependent robustness bound, (21), has been derived. A robust decentralized controller design approach using this bound has then been proposed. Apart from the derivation of the robustness bound, the proposed approach is decoupled for each subsystem and, for each subsystem, it is based on a local nominal model. Satisfying (26), however, assures the robust stability of the overall actual system under the designed controllers. Since the derived bound is frequency dependent, in general, the proposed approach is less conservative than an approach in which a constant bound is used. Furthermore, it also allows frequency shaping, [31].

Although, for brevity, we have considered only the case of a number of subsystems with a common overlapping part, the presented approach can also be extended to different cases of overlapping structures, such as the case of a string of overlapping subsystems, [15]. Another possible direction for future research is to extend the present approach to nonlinear and/or time-varying time-delay systems.

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The author has no conflicts of interest to declare that are relevant to the content of this article.