Consensus in Networks with Ostensible Metzler Agents

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Abstract: - This paper deals with consensus in a network of linear, ostensible Metzler agents whose communication topology is defined by an undirected graph. An observer-type consensus protocol is used and conditions for consensus within such a protocol are derived. The design problem is solved using separation of ostensible Metzler matrices and is parameterized using linear matrix inequalities. The methodology guarantees strictly positive observer and protocol gain matrices, which overcomes the conservatism generally caused by non-negative matrix parameters of agents. The properties of the proposed approach are illustrated by a numerical example.

Key-Words: - ostensible Metzler matrices, parametric constraints, linear matrix inequalities, state estimation, consensus protocol, cooperative systems control.

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1 Introduction

Cooperative systems control has gained considerable attention due to its applications in many fields, [1], [2], [3], [4], and has become one of the main principles in the control of multi-agent technical structures, focusing on the problems of consensus, coordination and synchronization. Using information about local neighbors, the main goal of the consensus task is to create a distributed algorithm (control protocol) to achieve the desired values of the state variables of all agents through information from a shared communication network, where each agent shares its local information with neighboring agents. The basic mathematical tools for consensus algorithms and their construction can be found, e.g., in [5], [6], [7]. For practical reasons, full consensus protocols in multiagent networks are replaced by protocols using distributed state estimation via observers, [8], [9], [10], where convergence of consensus algorithms requires weak connectivity of the communication graph, [11], [12]. Achieving consensus is a solution in a multiagent system, where the goal is for all agent states in the network to asymptotically converge to each other. Generalizations of these procedures, as well as possible application methods, can be found in [13], [14], [15].

A large class of multiagent systems consists of agents whose dynamics are defined by an ostensible Metzler matrix. These include networked multivariable flight systems, [16], embedded rocket fairing structures, [17], as well as underwater agent vehicles, [18] and represent primarily a class of externally positive systems. Their external positivity results in a restriction of synthesis methods for internally positive systems, [19], [20], which are strictly determined by the Metzler structure of dynamics, [21] and require consideration of the principle of diagonal stability.

The article deals with the analysis and synthesis of the foundations of consensus stability of identical linear ostensible Metzler agents using distributed estimation of the agent's state through a set of identical state observers. From the point of view of parametric constraints on Metzler matrices, a D-stability principle is proposed to construct a strictly negative and stable matrix using the negative elements of the ostensible Metzler matrix in such a way that the complement of this matrix to the original ostensible Metzler matrix is strictly a Metzler matrix. When considering multiagent systems composed of a set of N identical linear agents with strictly Metzler dynamics and a Laplace matrix for the topology of an undirected graph, the paper newly formulates sufficient conditions for the existence of consensus in a given closed-loop structure using singular value decomposition (SVD) of the Laplace matrix, which guarantee the solvability of the design problem while maintaining the required positivity. For this purpose, using the Lyapunov method and consensus asymptotic convergence constraints, conditions for the given problem are proposed in terms of linear matrix inequalities (LMI), which reflect the diagonal stability of the protocol and the parametric constraints of the agents. This concept does not take into account the presence of noise and disturbances.

The most commonly used scenario in distributed state estimation is one in which sensor measurements may be subject to approximately constant biases. In this case, the bias can be estimated using an extended state vector with variables representing the biases of the measurements. With such a formulation, hybrid Kalman filters, linear proportional-integral observers, and integral Kalman filters, [22], can be used to estimate the bias of all measurements. Since these structures are bound to discrete-time procedures, it is necessary to reformulate the presented problems for ostensible positive discrete-time models. Assuming that the observation errors and bias are independent of each other, it is reasonable to expect an approximate agreement in the values of the states of the graph nodes when the consensus protocol is violated by a bias. A possibility to extend the approach presented in this paper seems to be the use of minimum bias consensus with a substantial adaptation of the principles presented e.g. in [23], [24], which presents a topic for further research.

The outline of this paper is as follows. Basic introductory information is given in Section 2 and the consensus problem is solved in Section 3. The separation principle of ostensible Metzler matrices is given in Section 4 and LMIs for parameter design are formulated in Section 5. Section 6 demonstrates the applicability of the method with an example and Section 7 summarizes the results.

Throughout the article, $\boldsymbol{X} \prec 0$ denotes that the real matrix $\boldsymbol{X}$ is symmetric and negative definite, the notations $\boldsymbol{x}^{\mathrm{T}}$ ($\boldsymbol{X}^{\mathrm{T}}$) identify the transpose of the vector (matrix), $\boldsymbol{I}_n$ denotes the identity matrix of n-th order, $\mathrm{diag}[\cdot]$ enters the block diagonal matrix, the symbol $*$ represents the block induced by symmetry, $\otimes$ denotes the Kronecker matrix product, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times r}$, ($\mathbb{R}_+^{n\times r}$) refers to the set of (nonnegative) matrices of dimension $n \times r$, $\mathbb{R}_{-+}^{n\times n}$, ($\mathbb{R}_{-\oplus}^{n\times n}$) denotes the set of $n \times n$ strictly (ostensible) Metzler matrices.

2 Basic Preliminaries

Let $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$ be an undirected graph of order N, where $\mathcal{V} = \{1,\ldots,N\}$ is the set of nodes, $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}\}$ is the set of edges, and $\mathcal{A} = \{\alpha_{ij}\} \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix. The adjacency elements satisfy the conditions $\alpha_{ii} = 0$, $\alpha_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$. If the graph is undirected, $\alpha_{ij} = \alpha_{ji}$ and $\mathcal{A}$ is symmetric.

The Laplacian $\mathcal{L}$ for a graph $\mathcal{G}$ is defined as $\mathcal{L} = \{l_{ij}\} \in \mathbb{R}^{N\times N}$, $l_{ij} = -\alpha_{ij},\ i \neq j$ and $l_{ii} = \sum_{j\neq i} \alpha_{ij}$. For an undirected graph, both the adjacency matrix and the Laplacian are symmetric and the row and column sums of the Laplacian are zero, which means that for an undirected graph the Laplacian is singular and diagonally dominant, [25].

Kronecker product $\otimes$, which transforms two matrices $\boldsymbol{X} = \{x_{n,m}\} \in \mathbb{R}^{n\times m}$ and $\boldsymbol{Y} \in \mathbb{R}^{w\times v}$ into a larger matrix with the properties

$$\boldsymbol{X}\otimes\boldsymbol{Y} = \begin{bmatrix} x_{11}\boldsymbol{Y}\cdots x_{1m}\boldsymbol{Y} \\ \vdots \quad \ddots \quad \vdots \\ x_{n1}\boldsymbol{Y}\cdots x_{nm}\boldsymbol{Y} \end{bmatrix},\ \boldsymbol{X}\otimes\boldsymbol{Y} \in \mathbb{R}^{nw\times mv} \quad (1)$$

a satisfies the following basic properties for matrices $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}$ with the corresponding dimensions and scalar a, [26],

$$\begin{gathered} \boldsymbol{X}\otimes(\boldsymbol{Y}+\boldsymbol{Z}) = \boldsymbol{X}\otimes\boldsymbol{Y} + \boldsymbol{X}\otimes\boldsymbol{Z}, \\ (\boldsymbol{X}\otimes\boldsymbol{Z})(\boldsymbol{Y}\otimes\boldsymbol{W}) = \boldsymbol{X}\boldsymbol{Y}\otimes\boldsymbol{Y}\boldsymbol{W}, \\ (\boldsymbol{X}\otimes\boldsymbol{Z})^{\mathrm{T}} = \boldsymbol{X}^{\mathrm{T}}\otimes\boldsymbol{Z}^{\mathrm{T}}, \\ a\boldsymbol{X} = a\otimes\boldsymbol{X}, \end{gathered} \quad (2)$$

is used in this context to express the complete state-space model of all agents.

3 Consensus Problem

Consider multiagent systems with a set of N identical linear ostensible Metzler agents, described by realizations on the state space parameters $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ as

$$\dot{\boldsymbol{q}}_i(t) = \boldsymbol{A}\boldsymbol{q}_i(t) + \boldsymbol{B}\boldsymbol{u}_i(t), \quad \boldsymbol{y}_i(t) = \boldsymbol{C}\boldsymbol{q}_i(t), \quad (3)$$

where for $i = 1,\ldots,N$ vectors $\boldsymbol{q}_i(t) \in \mathbb{R}_+$, $\boldsymbol{u}_i(t) \in \mathbb{R}^r$, $\boldsymbol{y}_i(t)\mathbb{R}^m$, matrices $\boldsymbol{B}\mathbb{R}_+^{n\times r}$, $\boldsymbol{C}\mathbb{R}_+^{m\times n}$ are nonnegative and $\boldsymbol{A}\mathbb{M}_{-\oplus}^{n\times n}$ is an ostensible Metzler matrix.

For the i-th system (3), the observer takes the form

$$\begin{aligned} &\dot{\boldsymbol{q}}_{ei}(t) = \boldsymbol{A}\boldsymbol{q}_{ei}(t) + \boldsymbol{B}\boldsymbol{u}_i(t) + \boldsymbol{J}(\boldsymbol{y}_i(t) - \boldsymbol{y}_{ei}(t)), \\ &\boldsymbol{y}_{ei}(t) = \boldsymbol{C}\boldsymbol{q}_{ei}(t), \end{aligned} \quad (4)$$

where $\boldsymbol{q}_{ei}(t) \in \mathbb{R}^n$, $\boldsymbol{y}_{ei}(t) \in \mathbb{R}^m$ and $\boldsymbol{J} \in \mathbb{R}_+^{n\times m}$.

The task is to design a consensus protocol $\boldsymbol{u}_i(t)$ using the estimated values of the agents' state vectors such that

$$\boldsymbol{u}_i(t) = \boldsymbol{K}\sum_{j\in\mathcal{N}_\rangle} \alpha_{ij}(\boldsymbol{q}_{ei}(t) - \boldsymbol{q}_{ej}(t)), \quad (5)$$

where $\mathcal{N}_i$ is the neighborhood of the i-th agent, $\boldsymbol{K} \in \mathbb{R}_+^{r\times n}$ is the gain matrix and α_{ij} are the neighborhood elements of the matrix $\mathcal{A} = [\alpha_{ij}]_{i,j=1}^N \in \mathbb{R}^{N\times N}$.

Coupling the observer (4) and the system (3) as

$$\boldsymbol{e}_i(t) = \boldsymbol{q}_i(t) - \boldsymbol{q}_{ei}(t), \quad (6)$$

the dynamics of the error equation (6) is

$$\begin{aligned} \dot{\boldsymbol{e}}_i(t) &= \dot{\boldsymbol{q}}_i(t) - \dot{\boldsymbol{q}}_{ei}(t) \\ &= \boldsymbol{A}\boldsymbol{q}_i(t) + \boldsymbol{B}\boldsymbol{u}_i(t) - \\ &- \boldsymbol{A}\boldsymbol{q}_{ei}(t) - \boldsymbol{B}\boldsymbol{u}_i(t) - \boldsymbol{J}\boldsymbol{C}(\boldsymbol{q}_i(t) - \boldsymbol{q}_{ei}(t)) \\ &= (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_i(t) \end{aligned} \quad (7)$$

and, based on the Kronecker product properties (1), (2), then (7) gives

$$\begin{aligned} &\dot{\boldsymbol{e}}(t) = (\boldsymbol{I}_N \otimes \boldsymbol{A}_e)\boldsymbol{e}(t), \\ &\boldsymbol{e}(t) = \left[\boldsymbol{e}_1^{\mathrm{T}}(t)\cdots\boldsymbol{e}_N^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^{Nn}, \end{aligned} \quad (8)$$

where

$$\boldsymbol{A}_e = \boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}. \quad (9)$$

If the synchronizing state error vector is defined with the above introduced notations as

$$\boldsymbol{\delta}_i(t) = \boldsymbol{q}_i(t) - \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{q}_j(t) = \boldsymbol{q}_i(t) - \overline{\boldsymbol{q}}(t), \quad (10)$$

where it can be considered that

$$\overline{\boldsymbol{q}}(t) = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{q}_j(t), \quad (11)$$

then, using (5), (11), the time derivative of (10) is

$$\begin{aligned}\dot{\boldsymbol{\delta}}_i(t) &= \dot{\boldsymbol{q}}_i(t) - \dot{\overline{\boldsymbol{q}}}_i(t)\\ &= \boldsymbol{A}\boldsymbol{\delta}_i(t) + \boldsymbol{BK}\sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}(\boldsymbol{q}_{ei}(t) - \boldsymbol{q}_{ej}(t)) -\\ &- \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}\boldsymbol{BK}(\boldsymbol{q}_{ei}(t) - \boldsymbol{q}_{ej}(t))\\ &= \boldsymbol{A}\boldsymbol{\delta}_i(t) + \boldsymbol{BK}\sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}(\boldsymbol{q}_{ei}(t) - \boldsymbol{q}_{ej}(t)),\end{aligned} \quad (12)$$

whilst the final relation reflects symmetry of undirected topology graph $\alpha_{ij} = \alpha_{ji}$, which sets that

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_\rangle}\boldsymbol{BK}\alpha_{ij}(\boldsymbol{q}_{ei}(t) - \boldsymbol{q}_{ej}(t)) = \boldsymbol{0}.$$

Since it can be shown that

$$\begin{aligned}&\sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}(\boldsymbol{q}_{ei}(t) - \boldsymbol{q}_{ej}(t))\\ &= \sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}(\boldsymbol{q}_i(t) - \boldsymbol{e}_i(t) - (\boldsymbol{q}_j(t) - \boldsymbol{e}_j(t)))\\ &= \sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}\left(\boldsymbol{q}_i(t) - \overline{\boldsymbol{q}}(t) - \boldsymbol{q}_j(t)\right) +\\ &+ \sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}\left(\overline{\boldsymbol{q}}(t) - \boldsymbol{e}_i(t) + \boldsymbol{e}_j(t))\right)\\ &= \sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}\left(\boldsymbol{\delta}_i(t) - \boldsymbol{\delta}_j(t) - \boldsymbol{e}_i(t) + \boldsymbol{e}_j(t))\right),\end{aligned} \quad (13)$$

applying (13) then (12) can be written as

$$\begin{aligned}\dot{\boldsymbol{\delta}}_i(t) = \boldsymbol{A}\boldsymbol{\delta}_i(t) +&\\ + \boldsymbol{BK}&\sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}\left(\boldsymbol{\delta}_i(t) - \boldsymbol{\delta}_j(t))\right) -\\ - \boldsymbol{BK}&\sum_{j\in\mathcal{N}_\rangle}\alpha_{ij}\left(\boldsymbol{e}_i(t) - \boldsymbol{e}_j(t))\right)\end{aligned} \quad (14)$$

and by (14) the vector synchronizing error equation is

$$\begin{aligned}&\dot{\boldsymbol{\delta}}(t) = (\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{\delta}(t) + (\mathcal{L} \otimes \boldsymbol{BK})(\boldsymbol{\delta}(t) - \boldsymbol{e}(t)),\\ &\boldsymbol{\delta}(t) = \begin{bmatrix}\boldsymbol{\delta}_1^{\mathrm{T}}(t) & \cdots & \boldsymbol{\delta}_N^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{Nn}.\end{aligned} \quad (15)$$

The extended dynamics can now be rewritten using (8), (15) as

$$\begin{aligned}&\begin{bmatrix}\dot{\boldsymbol{\delta}}(t)\\ \dot{\boldsymbol{e}}(t)\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{I}_N \otimes \boldsymbol{A} + \mathcal{L} \otimes \boldsymbol{BK} & -\mathcal{L} \otimes \boldsymbol{BK}\\ \boldsymbol{0} & \boldsymbol{I}_N \otimes \boldsymbol{A}_e\end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}(t)\\ \boldsymbol{e}(t)\end{bmatrix}\end{aligned} \quad (16)$$

and accordingly it is observed from (16) that

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}^\circ\boldsymbol{z}(t), \quad (17)$$

where

$$\begin{aligned}&\boldsymbol{A}^\circ = \begin{bmatrix}\boldsymbol{I}_N \otimes \boldsymbol{A} + \mathcal{L} \otimes \boldsymbol{BK} & -\mathcal{L} \otimes \boldsymbol{BK}\\ \boldsymbol{0} & \boldsymbol{I}_N \otimes \boldsymbol{A}_e\end{bmatrix},\\ &\boldsymbol{z}^{\mathrm{T}}(t) = \begin{bmatrix}\boldsymbol{\delta}^{\mathrm{T}}(t) & \boldsymbol{e}^{\mathrm{T}}(t)\end{bmatrix}.\end{aligned} \quad (18)$$

Remark 1 *Since the Laplacian $\mathcal{L}$ of an undirected graph is a symmetric positive semi-definite matrix, the singular value decomposition of $\mathcal{L}$ of the form*

$$\boldsymbol{M}^{\mathrm{T}}\mathcal{L}\boldsymbol{M} = \boldsymbol{S}, \quad (19)$$

$$\boldsymbol{M} = [\boldsymbol{m}_1\, \boldsymbol{m}_2 \cdots \boldsymbol{m}_N],\ \boldsymbol{S} = \mathrm{diag}\,[s_1 \ldots s_{N-1}\, 0] \quad (20)$$

and with the singular value set written in an descending order in $\boldsymbol{S}$ as follows

$$s_1 \geq s_2 \geq \cdots \geq s_{N-1} \geq 0,$$

then the orthonormal vectors $\boldsymbol{m}_h$, $h = 1, \ldots, N$ from (19), (20) span the matrix $\boldsymbol{M}$ such that, [27],

$$\begin{aligned}&\boldsymbol{M}^{\mathrm{T}}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{M}^{\mathrm{T}} = \boldsymbol{I}_N,\ \boldsymbol{M}^{-1} = \boldsymbol{M}^{\mathrm{T}},\\ &\mathcal{L} = \boldsymbol{M}\boldsymbol{S}\boldsymbol{M}^{\mathrm{T}}.\end{aligned} \quad (21)$$

Theorem 1 *If for the closed-loop errors (17), (18), of the ostensible multi-agents (3) and for its Laplacian matrix $\mathcal{L}$ (19) there exists positive definite diagonal matrices $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathbb{R}^{n\times n}$ such that for $j = 1, \ldots, N-1$ the following set of LMIs is feasible*

$$\begin{matrix}\boldsymbol{P}_1 \succ 0,\\ \boldsymbol{P}_2 \succ 0,\end{matrix} \quad \Xi_j = \begin{bmatrix}\Xi_{j11} & *\\ -s_j\boldsymbol{P}_1\boldsymbol{BK} & \boldsymbol{\Theta}_{22}\end{bmatrix} \prec 0 \quad (22)$$

$$\boldsymbol{\Theta}_{22} = \boldsymbol{P}_2(\boldsymbol{A} - \boldsymbol{JC}) + (\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{J}^{\mathrm{T}})\boldsymbol{P}_2 \prec 0 \quad (23)$$

$$\boldsymbol{\Theta}_N = \boldsymbol{\Theta}_{22}, \quad \boldsymbol{\Sigma}_N = \boldsymbol{P}_1\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_1 \prec 0 \quad (24)$$

$$\Xi_{j11} = \boldsymbol{P}_1(\boldsymbol{A} + s_j\boldsymbol{BK}) + (\boldsymbol{A}^{\mathrm{T}} + s_j\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{P}_1 \quad (25)$$

then the consensus problem for (3) is solvable under the consensus protocol (5).

Hereafter, $*$ is the symmetric block matrix.

Proof: Respecting the principle of diagonal stabilization for the Metzler matrix structures, a candidate of the Lyapunov function for synchronizing error (17) is

$$v(\boldsymbol{z}(t)) = \boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{P}^\circ\boldsymbol{z}(t) > 0 \quad (26)$$

dictating for $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathbb{R}^{n\times n}$ that

$$\begin{aligned}&\boldsymbol{P}^\circ = \mathrm{diag}\,[\boldsymbol{I}_N \otimes \boldsymbol{P}_1\ \boldsymbol{I}_N \otimes \boldsymbol{P}_2] \succ 0,\\ &\boldsymbol{P}_1 = \mathrm{diag}\,[p_{11}\ \ldots\ p_{1n}] \succ 0,\\ &\boldsymbol{P}_2 = \mathrm{diag}\,[p_{21}\ \ldots\ p_{2n}] \succ 0.\end{aligned} \quad (27)$$

For $\boldsymbol{P}^\circ$ according to (27) the derivative $v(\boldsymbol{z}(t))$ along the solutions (26) is

$$\dot{v}(\boldsymbol{z}(t)) = \dot{\boldsymbol{z}}^{\mathrm{T}}(t)\boldsymbol{P}^\circ\boldsymbol{z}(t) + \boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{P}^\circ\dot{\boldsymbol{z}}(t) < 0. \quad (28)$$

Using the vector defined in (18), it follows from (28):

$$\begin{aligned}0 > &\begin{bmatrix}\dot{\boldsymbol{\delta}}^{\mathrm{T}}(t)\dot{\boldsymbol{e}}^{\mathrm{T}}(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{I}_N \otimes \boldsymbol{P}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_N \otimes \boldsymbol{P}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}(t)\\ \boldsymbol{e}(t)\end{bmatrix} +\\ &+ \begin{bmatrix}\boldsymbol{\delta}^{\mathrm{T}}(t)\,\boldsymbol{e}^{\mathrm{T}}(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{I}_N \otimes \boldsymbol{P}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_N \otimes \boldsymbol{P}_2\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{\delta}}(t)\\ \dot{\boldsymbol{e}}(t)\end{bmatrix}\end{aligned} \quad (29)$$

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and (29) can be represented as:

$$\begin{aligned}&\dot{\boldsymbol{\delta}}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_1)\boldsymbol{\delta}(t)+\boldsymbol{\delta}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_1)\dot{\boldsymbol{\delta}}(t)+\\&+\dot{\boldsymbol{e}}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_2)\boldsymbol{e}(t)+\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_2)\dot{\boldsymbol{e}}(t)\\&<0\,.\end{aligned}\tag{30}$$

This allows by using (8), (15), (21), to write

$$\begin{aligned}&\boldsymbol{\delta}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_1\boldsymbol{A}+\mathcal{L}\otimes\boldsymbol{P}_1\boldsymbol{BK})\boldsymbol{\delta}(t)+\\&+\boldsymbol{\delta}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_1\boldsymbol{A}+\mathcal{L}\otimes\boldsymbol{P}_1\boldsymbol{BK})^{\mathrm{T}}\boldsymbol{\delta}(t)+\\&+\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_2\boldsymbol{A}_e)\boldsymbol{e}(t)+\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_2\boldsymbol{A}_e))^{\mathrm{T}}\boldsymbol{e}(t)\\&-\boldsymbol{\delta}^{\mathrm{T}}(t)(\mathcal{L}\otimes\boldsymbol{P}_1\boldsymbol{BK})\boldsymbol{e}(t)-\boldsymbol{e}^{\mathrm{T}}(t)(\mathcal{L}\otimes\boldsymbol{P}_1\boldsymbol{BK})^{\mathrm{T}}\boldsymbol{\delta}(t),\end{aligned}\tag{31}$$

$$\begin{aligned}&\boldsymbol{\delta}^{\mathrm{T}}(t)(\boldsymbol{MM}^{\mathrm{T}}\otimes\boldsymbol{P}_1\boldsymbol{A}+\boldsymbol{MSM}^{\mathrm{T}}\otimes\boldsymbol{P}_1\boldsymbol{BK})\boldsymbol{\delta}(t)+\\&+\boldsymbol{\delta}^{\mathrm{T}}(t)(\boldsymbol{MM}^{\mathrm{T}}\otimes\boldsymbol{P}_1\boldsymbol{A}+\boldsymbol{MSM}^{\mathrm{T}}\otimes\boldsymbol{P}_1\boldsymbol{BK})^{\mathrm{T}}\boldsymbol{\delta}(t)\\&=\boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{N}^{\mathrm{T}}(\boldsymbol{I}_n\otimes\boldsymbol{P}_1\boldsymbol{A}+\boldsymbol{S}\otimes\boldsymbol{P}_1\boldsymbol{BK})\boldsymbol{N\delta}(t)+\\&+\boldsymbol{\delta}^{\mathrm{T}}(t)\boldsymbol{N}^{\mathrm{T}}(\boldsymbol{I}_n\otimes\boldsymbol{P}_1\boldsymbol{A}+\boldsymbol{S}\otimes\boldsymbol{P}_1\boldsymbol{BK})^{\mathrm{T}}\boldsymbol{N\delta}(t)\end{aligned}\tag{32}$$

and by defining the transformations $\boldsymbol{N}=\boldsymbol{M}^{\mathrm{T}}\otimes\boldsymbol{I}_N$

$$\begin{aligned}\boldsymbol{N\delta}(t)&=(\boldsymbol{M}^{\mathrm{T}}\otimes\boldsymbol{I}_N)\boldsymbol{\delta}(t)=\boldsymbol{\zeta}(t),\\\boldsymbol{Ne}(t)&=(\boldsymbol{M}^{\mathrm{T}}\otimes\boldsymbol{I}_N)\boldsymbol{e}(t)=\boldsymbol{\varepsilon}(t),\end{aligned}\tag{33}$$

(30) – (32) can be sequenced in the following relation

$$\begin{aligned}&\boldsymbol{\zeta}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_1\boldsymbol{A}+\boldsymbol{S}\otimes\boldsymbol{P}_1\boldsymbol{BK})\boldsymbol{\zeta}(t)+\\&+\boldsymbol{\zeta}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_1\boldsymbol{A}+\boldsymbol{S}\otimes\boldsymbol{P}_1\boldsymbol{BK})^{\mathrm{T}}\boldsymbol{\zeta}(t)+\\&+\boldsymbol{\varepsilon}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_2\boldsymbol{A}_e)\boldsymbol{\varepsilon}(t)+\boldsymbol{\varepsilon}^{\mathrm{T}}(t)(\boldsymbol{I}_N\otimes\boldsymbol{P}_2\boldsymbol{A}_e))^{\mathrm{T}}\boldsymbol{\varepsilon}(t)-\\&-\boldsymbol{\zeta}^{\mathrm{T}}(t)(\boldsymbol{S}\otimes\boldsymbol{P}_1\boldsymbol{BK})\boldsymbol{\varepsilon}(t)-\boldsymbol{\varepsilon}^{\mathrm{T}}(t)(\boldsymbol{S}\otimes\boldsymbol{P}_1\boldsymbol{BK})^{\mathrm{T}}\boldsymbol{\zeta}(t)\\&<0.\end{aligned}\tag{34}$$

Thus, in view of (20) and (33), (34), one can find that

$$\begin{aligned}&\sum_{j=1}^{N-1}\boldsymbol{\zeta}_j^{\mathrm{T}}(t)(\boldsymbol{P}_1(\boldsymbol{A}+s_j\boldsymbol{BK})\boldsymbol{\zeta}_j(t)+\\&+\sum_{j=1}^{N-1}\boldsymbol{\zeta}_j^{\mathrm{T}}(t)((\boldsymbol{A}+s_j\boldsymbol{BK})^{\mathrm{T}}\boldsymbol{P}_1)\boldsymbol{\zeta}_j(t)-\\&-\sum_{j=1}^{N-1}s_j(\boldsymbol{\zeta}_j^{\mathrm{T}}(t)\boldsymbol{P}_1\boldsymbol{BK}\boldsymbol{\varepsilon}_j(t)+\\&-\sum_{j=1}^{N-1}s_j(\boldsymbol{\varepsilon}_j^{\mathrm{T}}(t)\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{\zeta}_j(t))+\\&+\sum_{j=1}^{N}\boldsymbol{\varepsilon}_j^{\mathrm{T}}(t)(\boldsymbol{P}_2\boldsymbol{A}_e+\boldsymbol{A}_e^{\mathrm{T}}\boldsymbol{P}_2)\boldsymbol{\varepsilon}_j(t)\\&<0\end{aligned}\tag{35}$$

and connecting the variable vector components then (35) gives the stability condition

$$\begin{aligned}&\sum_{j=1}^{N-1}\boldsymbol{\xi}_j^{\mathrm{T}}(t)\Xi_j\boldsymbol{\xi}_j(t)+\\&+\boldsymbol{\varepsilon}_N^{\mathrm{T}}(t)\boldsymbol{\Theta}_N\boldsymbol{\varepsilon}_N(t)+\boldsymbol{\zeta}_N^{\mathrm{T}}(t)\boldsymbol{\Sigma}_N\boldsymbol{\zeta}_N(t)\\&<0\end{aligned}\tag{36}$$

where (36) is built on (22) – (24). ■

4 Ostensible Metzler Matrices

The matrix decomposition can be applied to an ostensible Metzler matrix $\boldsymbol{A}\in\mathbb{M}_{-\oplus}^{n\times n}$ in such a way that

$$\boldsymbol{A}=\boldsymbol{A}_p+\boldsymbol{A}_m,\tag{37}$$

where $\boldsymbol{A}_p\in\mathbb{M}_{-+}^{n\times n}$ is strictly Metzler and $\boldsymbol{A}_m\in\mathbb{R}_{-}^{n\times n}$ is element-wise negative and Hurwitz.

Lemma 1 [28] *For an ostensible Metzler $\boldsymbol{A}\in\mathbb{M}_{-\ominus}^{n\times n}$ and positive scalars $\eta,\delta\in\mathbb{R}_+$ there exists a strictly Metzler $\boldsymbol{A}_p\in\mathbb{M}_{-+}^{n\times n}$ and an element-wise negative and Hurwitz $\boldsymbol{A}_m\in\mathbb{R}_{-}^{n\times n}$ satisfying (37) if*

$$\boldsymbol{A}_p=(\boldsymbol{A}_d+\boldsymbol{A}^{+}+\eta\boldsymbol{\Sigma})+p\boldsymbol{I}_n=\boldsymbol{A}_p^{\circ}+p\boldsymbol{I}_n,\tag{38}$$

$$\boldsymbol{A}_m=(\boldsymbol{A}^{-}-\eta\boldsymbol{\Sigma})-p\boldsymbol{I}_n=\boldsymbol{A}_m^{\circ}-p\boldsymbol{I}_n,\tag{39}$$

$$\boldsymbol{A}_d=diag\left[-a_{11}-a_{22}\cdots-a_{nn}\right],\ p=\lambda_o+\delta,\tag{40}$$

$$\lambda_o=\max_k(\lambda_k^{+}|\lambda_k^{+}=\mathrm{real}(\lambda_k)>0);\lambda_k\in\rho(\boldsymbol{A}_m^{\circ}),\tag{41}$$

$$\begin{aligned}a_{ij}^{+}&=\begin{cases}a_{ij}\text{ if }a_{ij}>0,\\0\ \text{ if }a_{ij}<0,\end{cases}&&i\neq j,\\a_{ij}^{-}&=\begin{cases}a_{ij}\text{ if }a_{ij}<0,\\0\ \text{ if }a_{ij}>0,\end{cases}&&i,j\in[1,n],\end{aligned}\tag{42}$$

$$\boldsymbol{A}^{+}=\begin{bmatrix}0&a_{12}^{+}&\cdots&a_{1,n-1}^{+}&a_{1n}^{+}\\a_{21}^{+}&0&\cdots&a_{2,n-1}^{+}&a_{2n}^{+}\\\vdots&\vdots&\ddots&\vdots&\vdots\\a_{n-1,1}^{+}&a_{n-1,2}^{+}&\cdots&0&a_{n-1,n}^{+}\\a_{n1}^{+}&a_{n2}^{+}&\cdots&a_{n-1,n-1}^{+}&0\end{bmatrix},\tag{43}$$

$$\boldsymbol{A}^{-}=\begin{bmatrix}0&a_{12}^{-}&\cdots&a_{1,n-1}^{-}&a_{1n}^{-}\\a_{21}^{-}&0&\cdots&a_{2,n-1}^{-}&a_{2n}^{-}\\\vdots&\vdots&\ddots&\vdots&\vdots\\a_{n-1,1}^{-}&a_{n-1,2}^{-}&\cdots&0&a_{n-1,n}^{-}\\a_{n1}^{-}&a_{n2}^{-}&\cdots&a_{n-1,n-1}^{-}&0\end{bmatrix},\tag{44}$$

$$\boldsymbol{\Sigma}=\begin{bmatrix}0&1&\cdots&1&1\\1&0&\cdots&1&1\\\vdots&\vdots&\ddots&\vdots&\vdots\\1&1&\cdots&1&0\end{bmatrix},\tag{45}$$

where $\delta>0$ is tuning parameter for D-stability. Using (38) – (45), stability is reduced to the dynamics of $\boldsymbol{A}_p$.

5 Parameter Design

The main goal of this concept is to transform the synthesis form for ostensible Metzler matrices $\boldsymbol{A} \in \mathbb{M}_{-\ominus}^{n\times n}$ into the problem used for strictly Metzler matrices $\boldsymbol{A}_p \in \mathbb{M}_{-+}^{n\times n}$ while taking into account the principle of diagonal stabilization, defined in [29].

Lemma 2 [30] *If a strictly Metzler $\boldsymbol{A}_p \in \mathbb{R}_{-+}^{n\times n}$, $\boldsymbol{A}_p = \{a_{ij}\}_{i,j=1}^n$, is represented in the rhombic form*

$$\boldsymbol{A}_{\Theta p} = \begin{bmatrix} -a_{11} & & & \\ a_{21} & -a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & -a_{nn} \\ & a_{12} & \cdots & a_{1n} \\ & & \ddots & \vdots \\ & & & a_{n-1,n} \end{bmatrix}, \tag{46}$$

then the diagonal matrices, defined by the diagonals of (46) for $h = 0, \ldots, n-1$ as,

$$\begin{aligned} &\boldsymbol{A}_{\Theta p}(\upsilon,\upsilon) = \mathrm{diag}\left[-a_{1,1}\cdots -a_{n,n}\right] \prec 0, \\ &\boldsymbol{A}_{\Theta p}(\upsilon+h,\upsilon) \\ &= \mathrm{diag}\left[a_{1+h,1}\cdots a_{n,n-h}a_{1,n-h+1}\cdots a_{h,n}\right] \succ 0, \end{aligned} \tag{47}$$

reflect the strictly Metzler parametric constraints

$$a_{ii} < 0, \quad a_{ij} > 0,\ i \neq j,\ \forall\, i,j \in [1,n]. \tag{48}$$

Thus, it can set

$$\begin{aligned} &\boldsymbol{A}_{ep} = \boldsymbol{A}_p - \boldsymbol{J}\boldsymbol{C} \\ &\boldsymbol{C} = \begin{bmatrix} \boldsymbol{c}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c}_m^{\mathrm{T}} \end{bmatrix},\ \boldsymbol{C}_k = \mathrm{diag}\left[\boldsymbol{c}_k^{\mathrm{T}}\right] = \mathrm{diag}\left[c_{k1}\cdots c_{kn}\right] \\ &\boldsymbol{J} = [\boldsymbol{j}_1 \cdots \boldsymbol{j}_m],\ \boldsymbol{J}_k = \mathrm{diag}\left[\boldsymbol{j}_k\right] = \mathrm{diag}\left[j_{k1}\cdots j_{kn}\right] \end{aligned} \tag{49}$$

and the matrix $\boldsymbol{A}_{ep} \in \mathbb{M}_{-+}^{n\times n}$ is parameterized as

$$\begin{aligned} &\boldsymbol{A}_{ep} = \sum_{h=0}^{n-1} \boldsymbol{L}^h \Big(\boldsymbol{A}_{\Theta p}(\upsilon+h,\upsilon) - \sum_{k=0}^{m} \boldsymbol{J}_{kh}\boldsymbol{C}_k\Big), \\ &\boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-1} & \boldsymbol{0} \end{bmatrix} \in \mathbb{R}^{n\times n},\ \boldsymbol{J}_{kh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_k\boldsymbol{L}^h \end{aligned} \tag{50}$$

where $\boldsymbol{L}$ in (50) is the permutation matrix, [31].

Based on (47), (49), (50), the procedure is as follows:

Theorem 2 [32] *The observer (4) is stable with strictly Metzler and Hurwitz matrix $\boldsymbol{A}_{ep} \in \mathbb{M}_{-\ominus}^{n\times n}$ if for a strictly Metzler matrix $\boldsymbol{A}_p \in \mathbb{M}_{-+}^{n\times n}$ and nonnegative $\boldsymbol{C} \in \mathbb{R}_+^{m\times n}$ there exist positive definite diagonal matrices $\boldsymbol{P}_2, \boldsymbol{V}_k \in \mathbb{R}_+^{n\times n}$ such that for $h = 1, \ldots, n-1$, $k = 1, \ldots, m$, $\boldsymbol{l}^{\mathrm{T}} = [1, \cdots, 1] \in \mathbb{R}_+^n$ this set of LMIs is feasible*

$$\boldsymbol{P}_2 \succ 0, \quad \boldsymbol{V}_k \succ 0, \tag{51}$$

$$\boldsymbol{P}_2\boldsymbol{A}_{\Theta p}(\upsilon,\upsilon) - \sum_{k=1}^{m} \boldsymbol{V}_k\boldsymbol{C}_k \prec 0, \tag{52}$$

$$\boldsymbol{P}_2\boldsymbol{T}^h\boldsymbol{A}_{\Theta p}(\upsilon+h,\upsilon)\boldsymbol{T}^{h\mathrm{T}} - \sum_{k=1}^{m} \boldsymbol{V}_k\boldsymbol{T}^h\boldsymbol{C}_k\boldsymbol{T}^{h\mathrm{T}} \succ 0, \tag{53}$$

$$\boldsymbol{P}_2\boldsymbol{A}_p + \boldsymbol{A}_p^{\mathrm{T}}\boldsymbol{P}_2 - \sum_{k=1}^{m} \boldsymbol{V}_k\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_k - \sum_{k=1}^{m} \boldsymbol{C}_k\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{V}_k \prec 0. \tag{54}$$

Provided that the above conditions are met, the strictly positive observer gain matrix $\boldsymbol{J} \in \mathbb{R}_+^{n\times m}$ is constructed as

$$\boldsymbol{J}_k = \boldsymbol{P}_2^{-1}\boldsymbol{V}_k, \quad \boldsymbol{j}_k = \boldsymbol{J}_k\boldsymbol{l}, \quad \boldsymbol{J} = [\boldsymbol{j}_1 \cdots \boldsymbol{j}_m]. \tag{55}$$

Stabilising $\boldsymbol{\Theta}_{22}$ from (23), the separability principle application on (22) implies that $\Xi_{j11} \prec 0$ when considering (25).

Since (25) is bilinear with respect to $\boldsymbol{P}_1$ and $\boldsymbol{K}$, defining the matrix $\boldsymbol{Q}_1 = \boldsymbol{P}_1^{-1}$, the linearization can be achieved as follows:

$$\boldsymbol{Q}_1(\boldsymbol{P}_1(\boldsymbol{A}+s_j\boldsymbol{B}\boldsymbol{K}) + (\boldsymbol{A}^{\mathrm{T}}+s_j\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{P}_1)\boldsymbol{Q}_1 \prec 0 \tag{56}$$

and from (56) implies for $j = 1, \ldots, N-1$

$$\boldsymbol{A}_{cpj}\boldsymbol{Q}_1 + \boldsymbol{Q}_1\boldsymbol{A}_{cpj}^{\mathrm{T}} \prec 0,\ \boldsymbol{A}_{cpj} = \boldsymbol{A}_p + s_j\boldsymbol{B}\boldsymbol{K}. \tag{57}$$

Using the duality of controller and observer design, the parameterization for the consensus gain design problem is based on the following lemma:

Lemma 3 [30] *If a strictly Metzler matrix $\boldsymbol{A}_p \in \mathbb{R}_{-+}^{n\times n}$, $\boldsymbol{A}_p = \{a_{ij}\}_{i,j=1}^n$ is represented in the dual rhombic form*

$$\boldsymbol{A}_{\Theta} = \begin{bmatrix} -a_{11} & a_{12}\cdots & a_{1n} & & & \\ & -a_{22}\cdots & a_{2n} & a_{21} & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & -a_{nn} & a_{n1} & \cdots & a_{n,n-1} \end{bmatrix}, \tag{58}$$

then the diagonal matrices defined by the diagonals of (58) for $h = 0, \ldots, n-1$ as

$$\begin{aligned} &\boldsymbol{A}_{\Theta}(\upsilon,\upsilon) = \mathrm{diag}\left[-a_{1,1}\cdots -a_{n,n}\right] \prec 0, \\ &\boldsymbol{A}_{\Theta}(\upsilon,\upsilon+h) \\ &= \mathrm{diag}\left[a_{1,1+h}\cdots a_{n-h,n}a_{n-h+1,1}\cdots a_{n,h}\right] \succ 0, \end{aligned} \tag{59}$$

redefine the parametric constraints (48). Thus, it can set

$$\begin{aligned} &\boldsymbol{A}_{cpj} = \boldsymbol{A} + s_j\boldsymbol{B}\boldsymbol{K} \\ &\boldsymbol{K} = \begin{bmatrix} \boldsymbol{k}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{k}_m^{\mathrm{T}} \end{bmatrix},\ \boldsymbol{K}_k = \mathrm{diag}\left[\boldsymbol{k}_k^{\mathrm{T}}\right] = \mathrm{diag}\left[k_{k1}\cdots k_{kn}\right] \\ &\boldsymbol{B}_k = \mathrm{diag}\left[\boldsymbol{b}_k\right] = \mathrm{diag}\left[b_{k1} \ \cdots \ b_{kn}\right] \end{aligned} \tag{60}$$

and the matrix $\boldsymbol{A}_{cpj} \in \mathbb{M}_{-+}^{n\times n}$ is parameterized as

$$\begin{aligned} &\boldsymbol{A}_{cpj} = \sum_{h=0}^{n-1} \Big(\boldsymbol{A}_{\Theta p}(\upsilon,\upsilon+h) + s_j\sum_{k=0}^{m} \boldsymbol{B}_k\boldsymbol{K}_{kh}\Big)\boldsymbol{L}^h, \\ &\boldsymbol{K}_{kh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{K}_k\boldsymbol{L}^h. \end{aligned} \tag{61}$$

Based on (59)–(61), the procedure is as follows:

Theorem 3 *The consensus for (3) is stable with a strictly Metzler and Hurwitz matrix $\boldsymbol{A}_{pc} \in \mathbb{M}_{-+}^{n\times n}$ if for a strictly Metzler matrix $\boldsymbol{A}_p \in \mathbb{M}_{-+}^{n\times n}$ and a nonnegative $\boldsymbol{B} \in \mathbb{R}_{+}^{n\times r}$ there exist positive definite diagonal matrices $\boldsymbol{Q}_1, \boldsymbol{R}_k \in \mathbb{R}_{+}^{n\times n}$ such that for $h = 1,\ldots n-1$, $j = 1,\ldots,N-1$, $k = 1,\ldots,r$, $\boldsymbol{l}^{\mathrm{T}} = [1,\cdots,1] \in \mathbb{R}_{+}^{n}$ the feasibility of the following set of LMIs*

$$\boldsymbol{Q}_1 \succ 0,\ \boldsymbol{R}_k \succ 0, \tag{62}$$

$$\boldsymbol{A}_{\Theta p}(\upsilon,\upsilon)\boldsymbol{Q}_1 + s_j\sum_{k=1}^{m}\boldsymbol{B}_k\boldsymbol{R}_k \prec 0, \tag{63}$$

$$\boldsymbol{T}^h\boldsymbol{A}_{\Theta p}(\upsilon,\upsilon+h)\boldsymbol{T}^{h\mathrm{T}}\boldsymbol{Q}_1 + s_j\sum_{k=1}^{m}\boldsymbol{T}^h\boldsymbol{B}_k\boldsymbol{T}^{h\mathrm{T}}\boldsymbol{R}_k \succ 0, \tag{64}$$

$$\boldsymbol{A}_p\boldsymbol{Q}_1 + \boldsymbol{Q}_1\boldsymbol{A}_p^{\mathrm{T}} + s_j\sum_{k=0}^{m}(\boldsymbol{B}_k\boldsymbol{l}\,\boldsymbol{l}_n^{\mathrm{T}}\boldsymbol{R}_k + \boldsymbol{R}_k\boldsymbol{l}\,\boldsymbol{l}^{\mathrm{T}}\boldsymbol{B}_k) \prec 0. \tag{65}$$

implies a strictly positive gain matrix $\boldsymbol{K} \in \mathbb{R}_{+}^{n\times m}$ constructed as:

$$\boldsymbol{K}_k = \boldsymbol{R}_k\boldsymbol{Q}_1^{-1},\ \boldsymbol{k}_k^{\mathrm{T}} = \boldsymbol{l}_n^{\mathrm{T}}\boldsymbol{K}_k,\ \boldsymbol{K} = [\boldsymbol{k}_1\ \cdots\ \boldsymbol{k}_r]^{\mathrm{T}}. \tag{66}$$

The proof can be omitted because it also follows from the duality principle.

6 Illustrative Example

To present new results it is considered that the multi-agent structure is built on the linearized, continuous-time mathematical model of the F/A-18 Hornet aircraft [16],

$$\boldsymbol{A} = \begin{bmatrix}-1.1750 & 0.9871\\ -8.4580 & -0.9197\end{bmatrix},\ \rho(\boldsymbol{A}) = \begin{Bmatrix}-1.0474\\ \pm 2.8866\mathrm{i}\end{Bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix}0.19400 & 0.03593\\ 19.29000 & 3.80300\end{bmatrix},\quad \boldsymbol{C} = \boldsymbol{\Sigma} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

and it can be easily seen that this follows to start the decomposition of na ostensible Metzler matrix $\boldsymbol{A}$

$$\boldsymbol{A}_d = \mathrm{diag}\,[-1.1750\ \ -0.9197],$$

$$\boldsymbol{A}^{+} = \begin{bmatrix}0 & 0.9871\\ 0 & 0\end{bmatrix},\ \boldsymbol{A}^{-} = \begin{bmatrix}0 & 0\\ -8.458 & 0\end{bmatrix}.$$

Using the regularization value $\mu = 0.005$

$$\boldsymbol{A}_m^{\circ} = \begin{bmatrix}0 & -0.005\\ -8.463 & 0\end{bmatrix},\ \rho(\boldsymbol{A}_m^{\circ}) = \begin{Bmatrix}-0.2057\\ 0.2057\end{Bmatrix},$$

$$\lambda_o = \max_k\left(\lambda_k^{+}\,|\,\lambda_k^{+} = \mathrm{real}(\lambda_k) > 0, \lambda_k^{+} \in \rho(\boldsymbol{A}_m^{\circ})\right),$$

$$\lambda_o = 0.2057,$$

and choosing $\delta = 0.003$ as the parameter for setting the D-stability region of $\boldsymbol{A}_m$, then for given λ_0 the tuning parameter takes the value $p = 0.2087$ and this produces a negative and Hurwitz matrix $\boldsymbol{A}_m$ such that

$$\boldsymbol{A}_m = \begin{bmatrix}-0.2087 & -0.0050\\ -8.4630 & -0.2087\end{bmatrix},\quad \rho(\boldsymbol{A}_m = \begin{Bmatrix}-0.0030\\ -0.4144\end{Bmatrix}$$

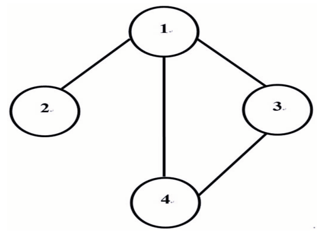

Fig. 1: Communication topology. Source: created by the author.

and a strictly Metzler matrix $\boldsymbol{A}_p = \boldsymbol{A}_d + \boldsymbol{A}^{+} + \eta\boldsymbol{\Sigma} + p\boldsymbol{I}_2$

$$\boldsymbol{A}_p = \begin{bmatrix}-0.9663 & 0.9921\\ 0.0050 & -0.7110\end{bmatrix},\ \rho(\boldsymbol{A}_p) = \begin{Bmatrix}-0.6929\\ -0.9844\end{Bmatrix}.$$

Used (47), (59), (49), (61) imply the parameters

$$\boldsymbol{A}_{\Theta}(\upsilon,\upsilon) = \mathrm{diag}\,[-0.9663\ \ -0.7110],$$

$$\boldsymbol{A}_{\Theta}(\upsilon+1,\upsilon) = \mathrm{diag}\,[0.0050\ \ 0.9921],$$

$$\boldsymbol{A}_{\Theta}(\upsilon,\upsilon+1) = \mathrm{diag}\,[0.9921\ \ 0.0050],$$

$$\boldsymbol{B}_1 = \mathrm{diag}\,[0.19400\ \ 19.29000],$$

$$\boldsymbol{B}_2 = \mathrm{diag}\,[0.03593\ \ 3.80300],$$

$$\boldsymbol{C}_1 = \mathrm{diag}\,[1\ \ 0],\quad \boldsymbol{C}_2 = \mathrm{diag}\,[0\ \ 1],$$

where for a network with the communication topology given in Figure 1 with $N = 4$, the Laplacian matrix and its singular values are

$$\mathcal{L} = \begin{bmatrix}3 & -1 & -1 & -1\\ -1 & 1 & 0 & 0\\ -1 & 0 & 2 & -1\\ -1 & 0 & -1 & 2\end{bmatrix},\ \boldsymbol{S} = \mathrm{diag}\,[4\ 3\ 1\ 0].$$

Using SeDuMi Toolbox, [33], it can be obtained the LMI variables from (51)–(54) as

$$\boldsymbol{P}_1 = \mathrm{diag}\,[0.9272\ 0.9007] \succ 0,$$

$$\boldsymbol{V}_1 = \mathrm{diag}\,[0.3971\ 0.4477] \succ 0,$$

$$\boldsymbol{V}_2 = \mathrm{diag}\,[0.0023\ 0.4256] \succ 0$$

and using (9), (49), (55), it results in:

$$\boldsymbol{J} = \begin{bmatrix}0.4284 & 0.0025\\ 0.4971 & 0.4725\end{bmatrix} > 0,$$

$$\boldsymbol{A}_e = \begin{bmatrix}-1.6918 & 0.4886\\ -8.4603 & -1.3018\end{bmatrix},$$

$$\varrho(\boldsymbol{A}_e) = \{-1.4968 \pm 2.0238\mathrm{i}\},$$

$$\boldsymbol{A}_{ep} = \begin{bmatrix}-1.4831 & 0.4936\\ 0.0027 & -1.0931\end{bmatrix},\ \varrho(\boldsymbol{A}_{ep}) = \begin{Bmatrix}-1.0897\\ -1.4865\end{Bmatrix}.$$

Solving (62)–(65), the LMI variables are as the diagonal matrices given below

$$\boldsymbol{Q}_1 = \mathrm{diag}\,[1.2619\ \ 0.9506] \succ 0,$$

$$\boldsymbol{R}_1 = \mathrm{diag}\,[0.0028\ \ 0.0007] \succ 0,$$

$$\boldsymbol{R}_2 = \mathrm{diag}\,[0.0202\ \ 0.0042] \succ 0.$$

and by (66), (57), it can be easily calculated that

$$\boldsymbol{K}=\begin{bmatrix}0.0022 & 0.0007\\ 0.0160 & 0.0045\end{bmatrix}>0$$

$$\boldsymbol{A}_{cp1}=\begin{bmatrix}-0.9623 & 0.9933\\ 0.4198 & -0.5878\end{bmatrix},\ \varrho(\boldsymbol{A}_{cp1})=\begin{Bmatrix}-0.1027\\ -1.4474\end{Bmatrix},$$

$$\boldsymbol{A}_{cp2}=\begin{bmatrix}-0.9633 & 0.9930\\ 0.3161 & -0.6186\end{bmatrix},\ \varrho(\boldsymbol{A}_{cp2})=\begin{Bmatrix}-0.2048\\ -1.3771\end{Bmatrix},$$

$$\boldsymbol{A}_{cp3}=\begin{bmatrix}-0.9653 & 0.9924\\ 0.1087 & -0.6802\end{bmatrix},\ \varrho(\boldsymbol{A}_{cp3})=\begin{Bmatrix}-0.4647\\ -1.1808\end{Bmatrix}.$$

The presented results confirm that with the defined Laplacian structure $\mathcal{L}$, the ostensible Metzler and Hurwitz matrix $\boldsymbol{A}$, and the non-negative matrices $\boldsymbol{B}$ and $\boldsymbol{C}$, the dynamics matrices of the observer and controller are strictly Metzler and Hurwitz.

7 Concluding Remarks

The subject of this paper is the design of a consensus protocol for a multi-agent system using estimated values of agent state vectors in a time-invariant communication topology, as derived from mathematical equations (1)(66). A distributed observer structure with shared sensors in a communication network is used, where an observer-based consensus control protocol is formulated for a set of identical agents described by linear ostensible Metzler mathematical models. Based on the author's original results in the field of separation of ostensible Metzler matrices, newly defined and verified conditions for consensus synthesis in a network of ostensible Metzler agents using SVD from the Laplacian are defined and verified so that the stability of the protocol is reflected in the optimization under given parametric constraints of the agents.

Respecting a given communication structure and the used control protocol, the joint effects of agent dynamics and communication graph on the consensus capability are initially formulated using the LMIs structure. Since consensus stability is defined only using the Lyapunov function, the design procedure has a separation property that is computationally advantageous. The methodology guarantees strictly positive observer and protocol gain matrices, which overcomes the conservatism generally caused by non-negative observer and protocol matrix parameters when solving the design scheme for non-negative agent matrix parameters.

Such a formulation of principles provides a potential basis for future research directions focused on more general cases of ostensible Metzler agents, such as discrete positive agents, distributed minimally biased consensus with applications in ostensible Metzler agents, and ostensible Metzler agents with time delays.

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