A Contribution on H_∞ Filter Design for Discrete-Time Linear Singular Systems with Parametric Uncertainties and Constant Time Delay

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Abstract: - This paper concerns an improved H_∞ filter design for discrete-time linear singular uncertain systems with constant time delay and unknown input. The proposed approach yields a robust observer that reconstructs a state functional despite the presence of uncertainties that are functions of a bounded noisy signal. The uncertain quantities are considered as disturbances. H_∞ filter is designed to attenuate the effect of parametric uncertainties on the estimation error and reconstructs a state functional independently of the actuator fault described by an unknown input vector. The proposed filter is based on the Krasovskii stability theory to ensure converging unbiased dynamics. The bias-free conditions are formulated into a convex optimization problem, leading to an optimal filter gain design. The results, validated on a benchmark numerical example and compared with existing methods, confirm the effectiveness of the proposed approach.

Key-Words: - Discrete transforms, Delay systems, H_∞ criterion, LMI, Singular systems, Uncertain systems.

Received: May 11, 2025. Revised: June 25, 2025. Accepted: August 27, 2025. Published: December 31, 2025.

1 Introduction

Monitoring a dynamic system for diagnostic purposes requires the knowledge of its state model, [1], [2], [3], [4]. However, in the case of singular linear uncertain systems, this task becomes more complex due to the differential and algebraic constraints of the system model, [5], [6], [7], [8].

Moreover, in practice, some of the state components are inaccessible which require the design of a state functional filter able to reconstruct components using measured outputs and known input, [9], [10], [11], [12]. This filter will minimize the impact of external perturbations and uncertainties on the state functional estimation error, [1], [13]. The linear matrix inequality (LMI) approach formalizes stability and robust performance constraints into a convex optimization problem, [14], [15]. For discrete systems, the synthesis of the filter relies on the minimization a criterion given by the ratio of the estimation error over the whole perturbation, [16]. This result offers flexibility to integrate additional constraints, such as convergence dynamic and robustness to structural variations.

Some previous works, [6], [16], [17], have dealt with the simultaneous handling of parameter uncertainties and state functional vector reconstruction where the uncertainties act on state only, [16], [17]. This work extends the previous ones to the case where the uncertainties and time delay are both acting on the state and the control input, [9].

Building on foundational work in functional state reconstruction for discrete-time singular systems, [11], this research extends previous developments, [16], [17], by addressing systems where uncertainties and time delays affect both state and control input simultaneously, [9], unlike earlier approaches focused solely on state uncertainties, [6]. The proposed discrete-time observer employs a bias-free estimation dynamic, with stability ensured through the discrete-time Lyapunov-Krasovskii theory formulated via LMIs, [14], [15]. This flexible LMI framework accommodates additional performance constraints.

The proposed algorithm consists on the determination of an observer gain that ensures bias-free estimation dynamics. This gain results from the solving of a set of LMIs developed from a discrete function equivalent to that of Lyapunov-Krasovskii. The paper is organized as follows: Section 2 formulates the modeling problem of delayed singular uncertain system. The third section outlines the assumptions and details the proposed algorithm. Section 4 validates the proposed algorithm on a numerical example based on a real benchmark.

2 Problem Formulation

The considered system (1 - 3) is a discrete singular uncertain and delayed system with an unknown input and where the uncertainty and the delay are operating on the state and the control input.

$$\begin{aligned} Ex(k+1) &= (A+\Delta A(k))x(k) + (B+\Delta B(k))u(k) \\ &+ (A_d+\Delta A_d(k))x(k-d) + +E_1 v(k) \\ &+ (B_d+\Delta B_d(k))u(k-d) \quad (1) \\ y(k) &= Cx(k) \quad (2) \\ z(k) &= Lx(k) \quad (3) \end{aligned}$$

where the vectors $x(k) \in \mathbb{R}^n$; $y(k) \in \mathbb{R}^p$; $u(k) \in \mathbb{R}^m$; $v(k) \in \mathbb{R}^r$ and $z(k) \in \mathbb{R}^q$ are respectively the state, the output, the input, the unknown input and the state functional at time instant k. The terms $\Delta A(k)$, $\Delta A_d(k) \in \mathbb{R}^{n*n}$ and $\Delta B(k)$, $\Delta B_d(k) \in \mathbb{R}^{n*m}$ correspond, respectively, to uncertainties on the matrices$A$, A_d, B and B_d.
d represents the delay and $E, A, A_d, B, B_d, E_1, C$ and L are constant matrices, with appropriate dimensions.

Hypothesis 1 The structured parametric uncertainties are defined as :

$$\begin{aligned} \Delta A(k) &= M_1 N_1 G(k) \\ \Delta A_d(k) &= M_2 N_2 G(k) \\ \Delta B(k) &= M_3 N_3 G(t) \\ \Delta B_d(k) &= M_4 N_4 G(k) \end{aligned} \quad (4)$$

Where M_i,$N_i (i = 1, ..., 4)$ are real constant matrices with appropriate dimensions and $G(k)$ is a bounded signal that satisfies

$$G(k)^\top G(k) \leq I \quad (5)$$

In the case where $G(k)$ is a scalar signal, inequality (5) simplifies to $|G(k)|^2 \leq 1$.

Hypothesis 2 Let's assume that, [16]:

1. $rank \begin{bmatrix} E \\ C \end{bmatrix} = n$
2. $rank\,[E] = \varepsilon_1 \leq n$

Under hypothesis 2, there exists a full-row-rank matrix $S \in \mathbb{R}^{(q+p)\times(n+p)}$ such that, [16]:

$$S = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad (6)$$

for which:

$$a_1 E + b_1 C = L_{q\times n} \quad (7)$$

$$c_1 E + d_1 C = 0_{p\times n} \quad (8)$$

with the matrices $a_1 \in \mathbb{R}^{q\times n}$, $b_1 \in \mathbb{R}^{q\times p}$, $c_1 \in \mathbb{R}^{p\times n}$, and $d_1 \in \mathbb{R}^{p\times p}$.

2.1 Discussion of Assumptions

Hypothesis 2, comprising the rank conditions rank$\begin{bmatrix} E \\ C \end{bmatrix} = n$ and rank$(E) = \varepsilon_1 \leq n$, ensures the existence of a solution matrix S for the functional estimation problem. These conditions are standard in singular system observation theory, [2], [16], and are typically satisfied in well-structured mechanical or electromechanical systems. If these conditions fail, alternative approaches such as weighted least-squares approximations could be considered, [2].

The bounded uncertainty structure $G(k)^\top G(k) \leq I$ in Hypothesis 1, while potentially conservative, is widely adopted in robust control to handle norm-bounded parametric variations, [7]. This formulation allows the uncertainty effect to be embedded into the LMI framework. The conservatism of this bound directly influences the achievable H_∞ performance level γ; a tighter bound generally allows for a smaller γ, corresponding to improved disturbance attenuation. The impact of this bound on the filter performance is analyzed in Section 5.

3 Discrete-Time filter design

In order to reconstruct the state functional of the system (1 - 3), the designed observer form is:

$$\begin{aligned} \Psi(k+1) &= F_0\Psi(k) + F_d\Psi(k-d) + H_0 u(k) \\ &+ H_d u(k-d) + L_1 y(k) \\ &+ L_2 y(k-d) \quad (9) \\ \hat{z}(t) &= \Psi(k) + b_1 y(k) + E_2 d_1 y(k) \quad (10) \end{aligned}$$

Figure 1 shows functional observer H_∞ Design.

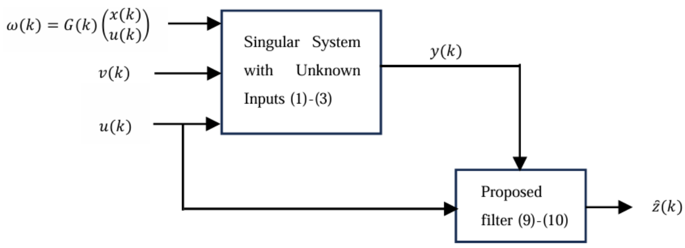

Fig. 1: Functional observer H_∞ design.
Source: Created by the authors.

Where $\Psi(k) \in \mathbb{R}^q$ represents the state functional vector of the observer, $\hat{z}(k)$ denotes the estimate of $z(t)$. The matrices F_0, F_d, H_0, H_d, L_1, L_2 and E_2 are matrices with appropriate dimensions that will be computed via LMI resolution to guarantee the asymptotic convergence from $\hat{z}(t)$ to $z(t)$, despite the presence of unknown inputs.

3.1 Filter Dynamic Scheme

The estimation error of state functional is:

$$e_z(k) = z(k) - \hat{z}(k) \tag{11}$$

By replacing $z(k)$ and $\hat{z}(k)$ by these expressions in (3) and (10), and using (2), (7) and (8), we have:

$$e_z(k) = WEx(k) - \Psi(k) \tag{12}$$

with

$$W = (a_0 + E_2 c_0) \tag{13}$$

From (11), the error $e_z(k+1)$ is given by:

$$e_z(k+1) = WEx(k+1) - \Psi(k+1) \tag{14}$$

By replacing $Ex(k+1)$ and $\Psi(k+1)$ by their expressions in (1) and (9) respectively we have:

$$\begin{aligned} e_z(k+1) = {} & (WA - F_0WE - L_1C)x(k) + F_0e_z(k) \\ & + (WB - H_0)u(k) + WE_1v(k) \\ & + (WB_d - H_d)u(k-d) + F_de_z(k-d) \\ & + (WA_d - F_dWE - L_2C)x(k-d) \\ & + \alpha\omega(k) + \beta\omega(k-d) \end{aligned} \tag{15}$$

with $\alpha = W[M_1N_1 \quad M_3N_3]$; $\beta = W[M_2N_2 \quad M_4N_4]$ and $\omega(k) = G(k)\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$.

Theorem 1 defines the observer (9 - 10) dynamics.

Theorem 1 The functional observer (9 - 10) acts as a state functional estimator for the system (1 - 3) if conditions i. to vi. are satisfied:

(i) $e_z(k+1) = F_0e_z(k) + F_de_z(k-d) + \alpha\omega(k) + \beta\omega(k-d)$ is asymptotically stable

(ii) $WA = F_0WE + L_1C$

(iii) $WA_d = F_dWE + L_2C$

(iv) $WB = H_0$

(v) $WB_d = H_d$

(vi) $WE_1 = 0$

△

proof 1 Starting from the expression of $e_z(k+1)$ (15) and substituting the uncertainty definitions from (4), we obtain

$$\begin{aligned} e_z(k+1) = {} & (WA - F_0WE - L_1C)x(k) + F_0e_z(k) \\ & + M_1N_1G(k)x(k) + M_3N_3G(k)u(k) \\ & + WE_1v(k) + (WB - H_0)u(k) \\ & + M_2N_2G(k)x(k-d) \\ & + M_4N_4G(k)u(k-d) + F_de_z(k-d) \\ & + (WA_d - F_dWE - L_2C)x(k-d) \\ & + (WB_d - H_d)u(k-d) \end{aligned} \tag{16}$$

Grouping the uncertainty terms by introducing $\omega(k) = G(k)\big[x(k)^\top \ \ u(k)^\top\big]^\top$, the above equation can be rewritten as

$$\begin{aligned} e_z(k+1) = {} & (WA - F_0WE - L_1C)x(k) + F_0e_z(k) \\ & + (WB - H_0)u(k) + WE_1v(k) \\ & + (WA_d - F_dWE - L_2C)x(k-d) \\ & + (WB_d - H_d)u(k-d) + \alpha\omega(k) \\ & + F_de_z(k-d) + \beta\omega(k-d) \end{aligned} \tag{17}$$

In order to have a bias-free form error such as:

$$\begin{aligned} e_z(k+1) = {} & F_0e_z(k) + F_de_z(k-d) \\ & + \alpha\omega(k) + \beta\omega(k-d) \end{aligned} \tag{18}$$

the coefficients multiplying $x(k)$, $u(k)$, $v(k)$, $x(k-d)$, and $u(k-d)$ must be zero. This leads directly to conditions(ii)-(vi) of Theorem 1. □

These conditions enable to compute the matrices of the observer, F_0, F_d, H_0, H_d, L_1, L_2 and E_2. When replacing W by its expression given by (13), conditions ii., iii. And vi. will be written as:

$$a_1A = F_0a_1E + J_0C - E_2c_1A \tag{19}$$

$$a_1A_d = F_da_1E + J_dC - E_2c_1A_d \tag{20}$$

$$a_1E_1 = -E_2c_1E_1 \tag{21}$$

with

$$J_0 = L_1 - F_0E_2d_1 \tag{22}$$

$$J_d = L_2 - F_dE_2d_1 \tag{23}$$

The equations (19 - 21) can be combined into matrices form as:

$$X\Gamma = \theta \tag{24}$$

where

$$X = [F_0 \quad F_d \quad J_0 \quad J_d \quad -E_2] \in \mathbb{R}^{q*(2q+3p)} \tag{25}$$

$$\Gamma = \begin{bmatrix} a_1E & 0 & 0 \\ 0 & a_0E & 0 \\ C & 0 & 0 \\ 0 & C & 0 \\ c_1A & c_1A_d & c_1E_1 \end{bmatrix} \in \mathbb{R}^{(2q+3p)*(2n+r)} \tag{26}$$

$$\theta = [a_1A \quad a_1A_d \quad a_1E_1] \in \mathbb{R}^{q*(2n+r)} \tag{27}$$

Since vector θ can be expressed as a linear combination of the last row of matrix Γ, it follows that:

$$rank\begin{pmatrix} \Gamma \\ \theta \end{pmatrix} = rank\,(\Gamma) \tag{28}$$

The solution of (24) is:

$$X = \theta\Gamma^+ - Z(I - \Gamma\Gamma^+) \tag{29}$$

where, Γ^+ denotes the pseudo inverse of matrix Γ. The matrix Z is determined using LMIs.
At this stage, we can deduce the matrices of the functional observer (9 - 10), F_0, F_d, J_0, J_d, L_1, L_2 and E_2.
The unknown matrix F_0 is defined as:

$$F_0 = X\alpha_1; \alpha_1 = \begin{bmatrix} I_q & 0_{(q+3p)*q} \end{bmatrix}^\top \tag{30}$$

By replacing (29) in (30), we have:

$$F_{011} = \theta\Gamma^+\alpha_1; F_{022} = (I - \Gamma\Gamma^+)\alpha_1 \tag{31}$$

The observer matrix F_0 is given by:

$$F_0 = F_{011} - ZF_{022} \tag{32}$$

and similarly, from the components of X:

$$F_d = F_{d11} - ZF_{d22} \tag{33}$$
$$J_0 = J_{011} - ZJ_{022} \tag{34}$$
$$J_d = J_{d11} - ZJ_{d22} \tag{35}$$
$$E_2 = E_{211} - ZE_{222} \tag{36}$$

where

$$F_{d11} = \theta\Gamma^+\alpha_2; F_{d22} = (I - \Gamma\Gamma^+)\alpha_2;$$
$$\alpha_2 = [0_{q*q} \quad I_q \quad 0_{3p*q}]^\top \tag{37}$$
$$J_{011} = \theta\Gamma^+\alpha_3; J_{022} = (I - \Gamma\Gamma^+)\alpha_3;$$
$$\alpha_3 = [0_{2q*p} \quad I_p \quad 0_{2p*p}]^\top \tag{38}$$
$$J_{d11} = \theta\Gamma^+\alpha_4; J_{d22} = (I - \Gamma\Gamma^+)\alpha_4;$$
$$\alpha_4 = [0_{3q*p} \quad I_p \quad 0_{p*p}]^\top \tag{39}$$
$$E_{211} = \theta\Gamma^+\alpha_5; E_{222} = (I - \Gamma\Gamma^+)\alpha_5;$$
$$\alpha_5 = \begin{bmatrix} 0_{(2q+2p)*p} & I_p \end{bmatrix}^\top \tag{40}$$

Then equation in condition i. from theorem 1 can be written such, [16]:

$$\begin{bmatrix} I_n & \zeta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_z(k+1) \\ \omega(k+1) \end{bmatrix} = \begin{bmatrix} F_0 & \alpha \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} e_z(k) \\ \omega(k) \end{bmatrix} + \begin{bmatrix} F_d & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_z(k-d) \\ \omega(k-d) \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} \omega(k) + K_y s(k) \tag{41}$$

In fact, we can write the perturbation dynamic as follow:

$$\omega(k+1) = K_x s(k) \tag{42}$$

with K_x is a given matrix with appropriate dimension and $s(k)$ is a random signal. Then multiplying (42) by a matrix ζ we have:

$$\zeta\omega(k+1) = \zeta K_x s(k)$$
$$\zeta\omega(k+1) = K_y s(k) \tag{43}$$

with $K_y = \zeta K_x$
In the sequel paper we will consider $\zeta = I$ and $K_y = I$ in order to simplify the mathematics.
Let's consider $\epsilon_z = \begin{bmatrix} e_z(k) \\ \omega(k) \end{bmatrix}$, then we have:

$$\rho\epsilon_z(k+1) = \hat{F}\epsilon_z(k) + \hat{F}_d\epsilon_z(k-d) + \hat{C}\hat{\omega}(k) \tag{44}$$

with

$$\rho = \begin{bmatrix} I_n & \zeta \\ 0 & 0 \end{bmatrix}; \hat{F} = \begin{bmatrix} F_0 & \alpha \\ 0 & -I_m \end{bmatrix}; \hat{F}_d = \begin{bmatrix} F_d & \beta \\ 0 & 0 \end{bmatrix};$$
$$\hat{C} = \begin{bmatrix} 0 & K_y \\ I_m & 0 \end{bmatrix}; \hat{\omega}(k) = \begin{bmatrix} \omega(k) \\ s(k) \end{bmatrix} \tag{45}$$

Based on (41 - 45), the observer (9 - 10) design is turned into a H_∞ filter development problem to attenuate perturbation $\omega(k)$ effect on the state functional estimation. The filter gain Z development procedure is described by theorem 2 based on a H_∞ criterion.

Theorem 2 The observer (9 - 10) design is equivalent to H_∞ filter development for the system (1 - 3) if matrices P, Q and Y exist, according to a H_∞ criterion such:

$$\frac{\|e_z(k)\|_\infty}{\|\omega(k)\|_\infty} < \gamma \tag{46}$$

So that the following optimization problem:

$$min(\gamma^2 \succ 0) \tag{47}$$

$$\rho^T P \rho = (\rho^T P \rho)^T \succ 0 \tag{48}$$

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_3 \end{bmatrix} \succ 0; P = P^T \tag{49}$$

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_4 \end{bmatrix} \succ 0; Q = Q^T \tag{50}$$

verifying

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} & \sigma_{17} & \sigma_{18} \\ * & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{28} \\ * & * & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} & \sigma_{37} & \sigma_{38} \\ * & * & * & \sigma_{44} & \sigma_{45} & \sigma_{46} & \sigma_{47} & \sigma_{48} \\ * & * & * & * & \sigma_{55} & \sigma_{56} & \sigma_{57} & \sigma_{58} \\ * & * & * & * & * & \sigma_{66} & \sigma_{67} & \sigma_{68} \\ * & * & * & * & * & * & \sigma_{77} & \sigma_{78} \\ * & * & * & * & * & * & * & \sigma_{88} \end{pmatrix} \prec 0 \tag{51}$$

with

$$\sigma_{11} = -P_1; \sigma_{12} = 0; \sigma_{13} = P_1 F_{011} - Y F_{022};$$
$$\sigma_{14} = P_1 \alpha_{011} - Y \alpha_{022}; \sigma_{15} = P_1 F_{d11} - Y F_{d22};$$
$$\sigma_{16} = P_1 \beta_{011} - Y \beta_{022}; \sigma_{17} = 0; \sigma_{18} = P_1 \quad (52)$$
$$\sigma_{22} = -P_3; \sigma_{23} = 0; \sigma_{24} = -P_3; \sigma_{25} = 0; \sigma_{26} = 0;$$
$$\sigma_{27} = P_3; \sigma_{28} = 0 \quad (53)$$
$$\sigma_{33} = Q_1 - P_1 + I; \sigma_{34} = -P_1; \sigma_{35} = 0; \sigma_{36} = 0;$$
$$\sigma_{37} = 0; \sigma_{38} = 0 \quad (54)$$
$$\sigma_{44} = Q_4 - P_1 + I; \sigma_{45} = 0; \sigma_{46} = 0; \sigma_{47} = 0;$$
$$\sigma_{48} = 0 \quad (55)$$
$$\sigma_{55} = -Q_1; \sigma_{56} = 0; \sigma_{57} = 0; \sigma_{58} = 0 \quad (56)$$
$$\sigma_{66} = -Q_4; \sigma_{67} = 0; \sigma_{77} = -\gamma^2 I_m; \sigma_{78} = 0;$$
$$\sigma_{88} = -\gamma^2 I_m \quad (57)$$

where

$$\alpha_{11} = (a_1 + (E_{211} \times c_1)) \times [M_1 N_1 \quad M_3 N_3] \quad (58)$$
$$\alpha_{22} = (E_{222} \times c_1) \times [M_1 N_1 \quad M_3 N_3] \quad (59)$$
$$\beta_{11} = (a_1 + (E_{211} \times c_1)) \times [M_2 N_2 \quad M_4 N_4] \quad (60)$$
$$\beta_{22} = (E_{222} \times c_1) \times [M_2 N_2 \quad M_4 N_4] \quad (61)$$

△

The gain Z is therefore given by:

$$Z = P^{-1} Y \quad (62)$$

proof 2 The proof is established using the Lyapunov-Krasovskii function in discrete time $V(\epsilon_z, k)$, [14], [16], which serves as an energy-like function for the estimation error. To further address disturbance attenuation, H_∞ criterion is embedded into the functional's structure, $V(\epsilon_z, k)$ describing the estimation dynamic error. Then we propose:

$$V(\epsilon_z, k) = \epsilon_z^T(k) \rho^T P \rho \epsilon_z(k) + \sum_{i=k-d}^{k-1} \epsilon_z^T(i) Q \epsilon_z(i) \quad (63)$$

To establish a sufficient condition for the existence of (9 - 10), where P and Q satisfy (49) and (50), we propose to verify the following condition:

$$H(\epsilon_z, \omega) = \Delta V(\epsilon_z, k) + \epsilon_z^T(k) \epsilon_z(k) - \gamma^2 \omega^T(k) \omega(k) \prec 0 \quad (64)$$

with

$$\Delta V(\epsilon_z, k) = V(k+1) - V(k) \quad (65)$$

$$\begin{aligned} H(\epsilon_z, \omega) = {} & \epsilon_z^T(k)[\hat{F}^T P \hat{F} + Q - \rho^T P \rho + I] \epsilon_z(k) \\ & + \epsilon_z^T(k-d)[\hat{F}_d^{\,T} P \hat{F}_d - Q] \epsilon_z(k-d) \\ & + \epsilon_z^T(k) \hat{F}^T P \hat{F}_d \epsilon_z(k-d) \\ & + \epsilon_z^T(k) \hat{F}^T P \hat{C} \omega(k) \\ & + \epsilon_z^T(k-d) \hat{F}_d^{\,T} P \hat{F} \epsilon_z(k) \\ & + \epsilon_z^T(k-d) \hat{F}_d^{\,T} P \hat{C} \omega(k) \\ & + \omega^T(k) \hat{C}^T P \hat{F} \epsilon_z(k) \\ & + \omega^T(k) \hat{C}^T P \hat{F}_d \epsilon_z(k-d) \\ & + \omega^T(k)(\hat{C}^T P \hat{C} - \gamma^2 I_m) \omega(k) \prec 0 \end{aligned} \quad (66)$$

which can be written as follows:

$$\Phi^T(k) \eta \Phi(k) \quad (67)$$

with

$$\eta = \begin{bmatrix} \hat{F}^T P \hat{F} + Q - \rho^T P \rho + I & \hat{F}^T P \hat{F}_d & a \\ * & \hat{F}_d^{\,T} P \hat{F}_d - Q & b \\ * & * & c \end{bmatrix} \quad (68)$$

with $a = \hat{F}^T P \hat{C}$, $b = \hat{F}_d^{\,T} P \hat{C}$ and $c = \hat{C}^T P \hat{C} - \gamma^2 I_m$.
and

$$\Phi(k) = \begin{bmatrix} \epsilon_z(k) \\ \epsilon_z(k-d) \\ \omega(k) \end{bmatrix} \quad (69)$$

To eliminate the quadratic form present in the inequality (68), we reformulate it as follows:

$$\Phi(k) = \xi - \phi^T \Xi^{-1} \phi \quad (70)$$

with

$$\xi = \begin{bmatrix} Q - \rho^T P \rho + I & 0 & 0 \\ * & -Q & 0 \\ * & * & -\gamma^2 I_m \end{bmatrix} \quad (71)$$

$$\phi = \begin{bmatrix} \hat{F} & \hat{F}_d & \hat{C} \end{bmatrix} \quad (72)$$

and

$$\Xi = -P^{-1} \quad (73)$$

Based on Schur's lemma, [18], the conditions $\eta \prec 0$ and $\Xi \prec 0$ hold if and only if:

$$\Lambda = \begin{bmatrix} \Xi & \phi \\ \phi^T & \Phi \end{bmatrix} \prec 0 \quad (74)$$

Using a congruence transformation, [19], we transform Λ as follows:

$$\upsilon^T \Lambda \upsilon \prec 0 \quad (75)$$

where

$$\upsilon = \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad (76)$$

Replacing Λ and υ respectively by their expressions (74) and (76) and using (71 - 76), (70) is equivalent to:

$$\upsilon = \begin{bmatrix} -P & P\hat{F} & P\hat{F}_d & P\hat{C} \\ * & Q - \rho^T P \rho + I & 0 & 0 \\ * & * & -Q & 0 \\ * & * & * & -\gamma^2 I_m \end{bmatrix} \prec 0 \quad (77)$$

Then, replacing in (77) F_0 and F_d , which appear in $\hat{F}$ and $\hat{F}_d$ respectively, by their expressions (32) and (33) respectively, theorem 2 is proven.

□

3.2 Algorithm for Functional Filter Design

Given a system defined by model (1 - 3), we then proceed to determine the temporal functional observer by applying the algorithm to a discrete-time numerical simulation.

1. Check those conditions 1) and 2) of hypothesis 2 are satisfied;
2. Compute matrix S given by (6) using (7) and (8);
3. Compute matrices Γ and θ using (26) and (27);
4. Determine matrices F_{011}, F_{022}, F_{d11}, F_{d22}, J_{011}, J_{022}, J_{d11}, J_{d22}, E_{211} and E_{122} from (31) and (37) and (40);
5. Calculate the observer gain Z according to (62);
6. Compute F_0, F_d, J_0, J_d and E_2 from the equations (32 - 36);
7. Calculate L_1 and L_2 from (22) and (23);
8. Determine the matrices H and H_d in conditions (iv) and (v) of as mentioned theorem 1.

4 Numerical Example

The process treated in this paragraph is a Marble Bloc Reverser, [20], whose mechanical configuration is illustrated in Figure 2 and Figure 3. Figure 2 shows the first mobile plate, controlled in on/off mode by a hydraulic jack. Figure 3 depicts the second reversing plate, controlled by a hydraulic energy distribution system composed of a hydraulic jack whose input flow is regulated by a DC servo-motor, [21].

$$H(s) = \frac{K_T}{J_i R_a s^2 + (B_{vf} R_a + K_T K_{f.e.m}) s} \quad (78)$$

with K_T, J_i, B_{vf} and $K_{f.e.m}$ are respectively, the torque constant, the inertia constant, the armature resistance, the viscosity-friction coefficient and the constant of the electromotive force (f.e.m).

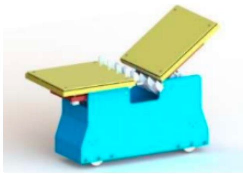

Fig. 2: First mobile plate of the block tipper.
Source: [20].

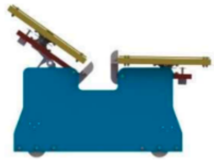

Fig. 3: Second reversing plate of the block.
Source: [20].

The continues state equation of the process is:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (79)$$
$$y(t) = Cx(t) \quad (80)$$

with $A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{Z_a}{J_i R_a} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{K_T}{J_i R_a} \end{bmatrix}$, $Z_a = 24$, $J_i = 1$, $R_a = 4$ and $K_T = 2$.

We assume that this process may contains uncertainties and time delay and it can be supplied by an unknow unput so that we adopt the process model to our model (1 - 3). The correspond discrete model is obtained through sampling.
The matrices A and B of model (1 - 3) are chosen that of the process and the other matrices are set as:

$$E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; A = \begin{pmatrix} 0 & 1 \\ 0 & -6 \end{pmatrix}; A_d = \begin{pmatrix} -0.23 & 0 \\ 0 & -0.7 \end{pmatrix};$$
$$B = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}; B_d = \begin{pmatrix} 5 \\ 2 \end{pmatrix}; E_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}; C = \begin{pmatrix} 1 & 1 \end{pmatrix};$$
$$L = \begin{pmatrix} 1 & 2 \end{pmatrix}; M_1 = \begin{pmatrix} 0.25 & 0.1 \\ 2 & -0.1 \end{pmatrix}; M_2 = \begin{pmatrix} 0.5 & -1 \\ 2.1 & 1 \end{pmatrix};$$
$$M_3 = \begin{pmatrix} 0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix}; M_4 = \begin{pmatrix} 0.1 & 0.1 \\ 2 & -0.1 \end{pmatrix};$$
$$N_1 = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix}; N_2 = \begin{pmatrix} 0.01 & 0.65 \\ 0 & 0.5 \end{pmatrix};$$
$$N_4 = \begin{pmatrix} 0.65 \\ 0.4 \end{pmatrix} \quad \text{and} \quad N_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (81)$$

The time delay is set to $d = 1s$.
The discrete-time control input $u(k)$, the unknown input $v(k)$ and the uncertainty signal $G(k)$ affecting the system are represented respectively in Figure 4, Figure 5 and Figure 6.
In this configuration, signal $G(k)$ fulfills condition in (7) of hypothesis 2.

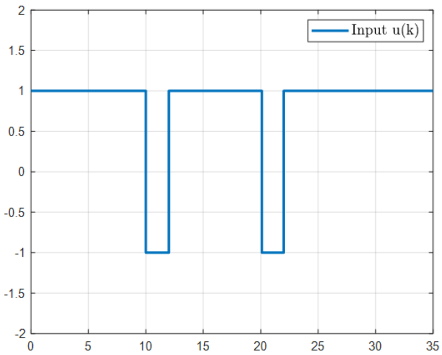

Fig. 4: Input u(k).
Source: Created by the authors.

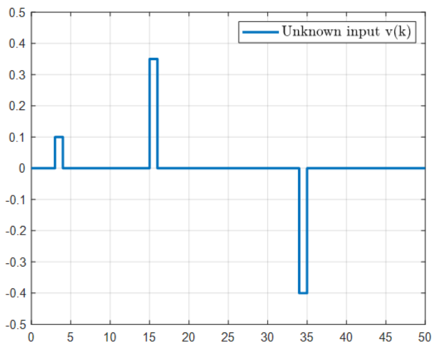

Fig. 5: Unknown input v(k).
Source: Created by the authors.

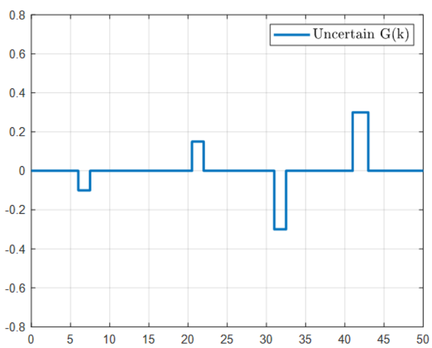

Fig. 6: Uncertain G(k).
Source: Created by the authors.

According to assumptions (7) and (8), the matrices a_1; b_1; c_1 and d_1 are defined as follows:

$$\begin{aligned} a_1 &= \begin{bmatrix} 0 & -1 \end{bmatrix}; b_1 = 2; \\ c_1 &= \begin{bmatrix} 1 & -1 \end{bmatrix} \quad \text{and} \quad d_1 = 0. \end{aligned} \tag{82}$$

The discrete-time LMI-based approach determines observer matrices that guarantee stability, robustness, and convergence of estimation errors under uncertainties, time delays, and discrete-time dynamics, while optimizing performance through H_∞ criteria.

$$\begin{aligned} &F_0 = -29; F_d = -3.95; H_0 = 2; H_d = -17; \\ &L_1 = -29; L_2 = -2.8 \quad \text{and} \quad E_2 = -5. \end{aligned} \tag{83}$$

Figure 7 shows the discrete-time evolution of the state components $x_1(k)$ and $x_2(k)$, highlighting their dynamic behavior and mutual interactions over the observation period.

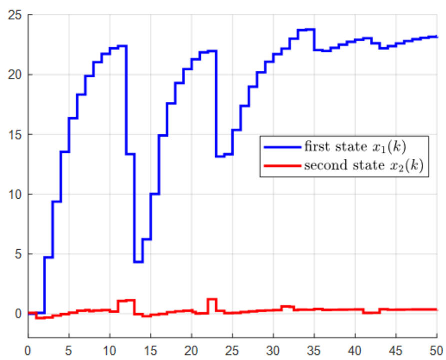

Fig. 7: Evolution of components $x\ (k)$.
Source: Created by the authors.

Figure 8 shows the real and estimated values of the functional state vector $z(k)$, highlighting the performance of the proposed observer in accurately tracking the system's dynamic evolution. The comparison between the actual functional state signal and its estimate demonstrates the effectiveness of the filtering approach.

To provide a more detailed insight into the estimation accuracy, Figure 9 presents two zoomed views corresponding to the time intervals $[12.5, 14.5]$ and $[40, 45]$, covering both transient and steady-state phases. These detailed views reveal that the estimation error remains consistently low, confirming the robustness and reliability of the adopted observation methodology. The improved tracking accuracy of the proposed observer is clearly visible when comparing the detailed zoom in Figure 9 with the overall view in Figure 8.

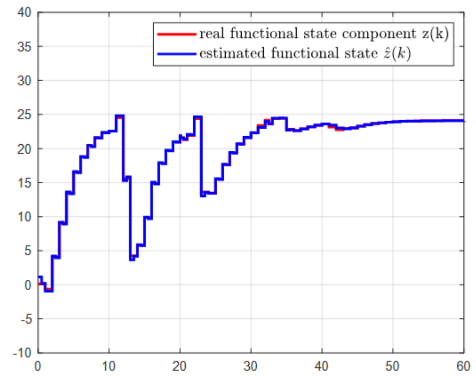

Fig. 8: Component of $z\ (k)$: real $z(k)$ and estimated $\hat{z}(k)$.
Source: Created by the authors.

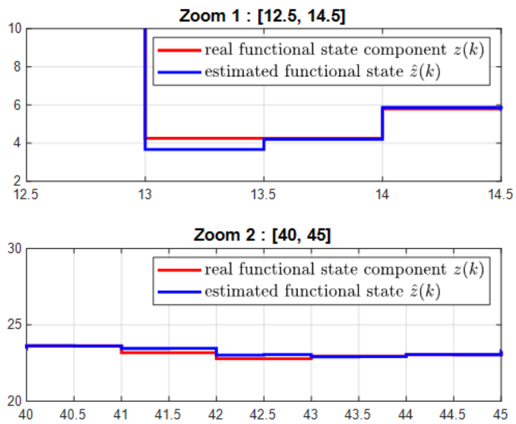

Fig. 9: Zoomed Analysis of Signals $z\ (k)$ and $\hat{z}(k)$ Over Two Time Intervals.
Source: Created by the authors.

In the discrete-time framework, the estimated state functional $\hat{z}(k)$ converges to the real values $z(k)$, even in presence of parametric uncertainties and unknown input. This robustness stems from the state functional estimating approach, which attenuates bounded uncertainties and compensates for unknown inputs. As shown in Figure 10, the estimation errors converge toward zero over time, confirming the algorithm's stability and effectiveness. The convergence is further supported by theoretical guarantees, ensuring consistent performance in dynamic environments.

The optimal H_∞ performance level is $\gamma = 0.45$. The proposed observer outperforms both the approach in [16], and a standard Luenberger observer, [22], offering better disturbance attenuation.

The estimation accuracy is further reflected in the convergence of the estimation error $e_z(k)$ to zero, as shown in Figure 10. This demonstrates the practical effectiveness of the proposed observer and its improved performance compared to [16], and the standard Luenberger observer, [22].

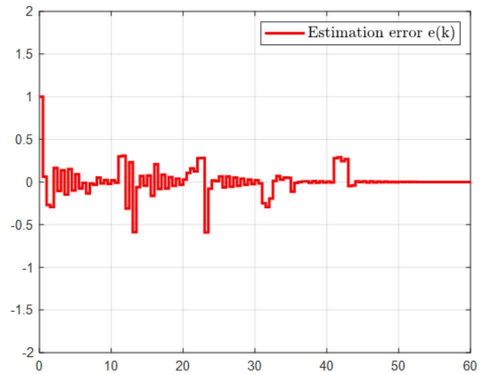

Fig. 10: Estimation error e(k)
Source: Created by the authors.

5 Conclusion

This work proposes a H_∞ filter for discrete-time singular systems with parametric uncertainties, delays, and unknown inputs. By structuring uncertainties as bounded disturbances, the proposed method attenuates their impact via LMI-based convex optimization, while reconstructing functional states independently of faults. Based on Lyapunov-Krasovskii theory, stability conditions ensure convergent error dynamics and bounds the energy gain between disturbances and estimation errors. The established LMIs guarantee both asymptotic stability and prescribed disturbance attenuation levels for the estimation error dynamics. Through a numerical example, the approach demonstrates robustness in under-instrumented or constrained contexts, showing effective state reconstruction despite complex system dynamics and measurement limitations.

As derived from the mathematical equations (1)–(83), the proposed design framework rigorously establishes the stability and H_∞ performance conditions of the estimation error system.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)
The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflicts of Interest
The authors have no conflicts of interest to declare.