

Robust Recursive Least-Squares Predictor with Covariance Information in Linear Continuous-Time Stochastic Systems with Uncertainties

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Abstract: - This paper presents, in Theorem 1, the robust recursive least-squares (RLS) prediction and filtering algorithm for the signal and the nominal system state (SS). The algorithm uses cross-covariance (CC) information between the nominal and degraded SS, as well as auto-covariance (AC) information for the degraded SS, both expressed in semi-degenerate kernel forms. The algorithm also utilizes the observation matrices for the system and the degraded system, as well as the variance of the white Gaussian observation noise (WGON). A recursive algorithm for the prediction error variance function of the robust RLS predictor in Theorem 1 is presented. Based on the robust RLS prediction and filtering algorithm in Theorem 1, Theorem 2 presents the robust RLS Wiener prediction and filtering algorithm. This algorithm utilizes the corresponding system and observation matrices in both nominal and degraded systems, the cross-variance (CV) function between the nominal SS and the degraded SS, the auto-variance (AV) function of the degraded SS, and the variance of the WGON. A numerical simulation example illustrates the efficacy of the robust RLS prediction and filtering algorithm proposed in Theorem 1. The proposed robust predictor and filter demonstrate improved estimation accuracy compared to the existing RLS predictor and filter, which use covariance information.

Key-Words: - Uncertain parameters, robust predictor, initial-value problem, Volterra-type integral equation, continuous-time uncertain systems, semi-degenerate kernel.

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1 Introduction

For uncertain systems, researchers have devised robust estimation techniques in continuous-time and discrete-time stochastic systems with uncertainties [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]. In [1], the authors investigate the H_∞ robust filtering problem for uncertain finite-frequency continuous-time systems. In [2], the researchers developed a robust state estimation approach for power systems with non-Gaussian noise statistics, based on an L_p -norm-based estimator and a cubic Kalman filter. [3] introduces a switching Gaussian heavy-tailed distribution for nonlinear systems, incorporating non-stationary heavy-tailed processes along with measurement noise. This paper derives a novel switching Gaussian-heavy-tailed distribution from a robust fixed-interval smoother based on the Student's distribution and variational Bayesian inference. In [4], the authors investigate the stability of filters for controlled stochastic dynamical systems, providing conditions under which a mis-initialized filter will

converge to the same state as a correctly initialized one. The author designs an original robust RLS finite impulse response filter for linear continuous-time systems with uncertainties in both the system and observation matrices [5]. This study employs an observable companion form to represent the state-space model of the degraded signal. A λ -contaminated framework, which does not require an explicit model of the uncertainties, is used in [6] to examine the filtering and fixed-lag smoothing for uncertain discrete-time systems. The results are formulated in terms of transfer functions and polynomials, employing two spectral factors together with a Diophantine equation to derive the estimator. The robust multi-sensor multi-target tracking algorithm presented in [7] is built upon a low-cost, robust cardinalized probability hypothesis density filter for clutter-rate estimation and a multi-sensor generalized labeled multi-Bernoulli filter for multi-target tracking. In Chapter 9, authors in [8] discuss estimating the H_∞ finite impulse response state for uncertain systems. In the event of a modeling

mismatch between the provided nominal nonlinear state-space model and the real one, authors in [9] suggest a robust extended Kalman smoothing method. Authors in [10] suggest a robust extended Kalman filter combined with a measurement compensation technique, where gross error identification is carried out within a moving window and anomalous measurement data is detected using a statistical test method. The gross error magnitudes are then calculated and used to measure compensation. Finally, the extended Kalman filter is used to update the compensated measurements and re-estimate the accurate states. A robust filtering problem is considered for a class of nonlinear systems. Affine fuzzy parts and norm-bounded uncertainties describe these systems [11]. In [12], the authors present a robust Kalman filter and a robust predictor for uncertain systems affected by mixed uncertainties, including uncertain-variance multiplicative noise and missing measurements. The paper in [13] develops a robust fusion time-varying Kalman filter and predictor for multi-sensor networked systems subject to uncertainties, including multiplicative noise and missing measurements. The study in [14] investigates game-theoretic solutions for robust linear filters and predictors in the context of uncertain system dynamics and initial state conditions. The study focuses on M-TH order time-varying discrete-time systems that incorporate uncertain input and observation noise. The Robust Centralized and Weighted Observation Fusion (CAWOF) prediction algorithm [15] has been developed for use in uncertain multi-sensor systems. It explicitly accounts for linear correlations between observations and input noises, as well as for uncertainties in variance and the effects of multiplicative noise. The robustness of the two fusion predictors is demonstrated through Lyapunov equation-based analysis, which shows that the CAWOF predictors achieve minimum upper bounds on prediction error variance. Authors in [16] address the challenge of designing robust predictors for stochastic dynamical systems that incorporate uncertainty quantification. Traditional robust predictors primarily focus on improving point predictions and therefore struggle to provide reliable UQ. To overcome this limitation, the authors propose a method that enhances probabilistic predictions for SDSs, thereby enabling robust distributional UQ. The paper in [17] addresses the problem of robust weighted state-fusion estimation in time-varying multi-sensor networked systems subject to mixed uncertainties. These uncertainties include uncertain-variance multiplicative noise, linearly correlated additive white noise, and packet dropouts.

By employing the augmented-state method together with the fictitious-noise technique, the system is reformulated so that only the noise-variance uncertainties remain to be handled. Based on the minimax robust estimation principle, the filter, the predictor, and the smoother are derived. The paper in [18] investigates a robust Kalman predictor for descriptor systems subject to multiple sources of uncertainty, including noise-variance fluctuations, random packet losses, and measurement delays. Singular value decomposition is employed to reduce the descriptor system, while information loss caused by unreliable communication networks is mitigated by augmenting the state and introducing fictitious noises. These steps convert the original descriptor model into a standard state-space form. A robust predictor is then derived based on the minimax robust estimation principle, ensuring minimal filtering-error variance under unknown noise variances. This paper adopts an approach of expressing the degraded system in the state-space model in observable companion form. Among these types of estimators, the focus is on the robust predictor. Theorem 1 presents a robust RLS prediction and filtering algorithm for both the signal and the nominal system state (SS) in linear, continuous-time, uncertain, stochastic systems. The algorithm uses the CC information between the nominal SS and the degraded SS, as well as the AC information of the degraded SS $\tilde{x}(t)$, in both semi-degenerate kernel forms. The algorithm also utilizes the observation matrices for the system and the degraded system, as well as the variance of the white Gaussian observation noise (WGON). The authors in [19] describe the semi-degenerate kernel within the context of a Fredholm integral equation. Theorem 2 then presents the robust RLS Wiener prediction and filtering algorithm for both the signal and the SS. The algorithm utilizes the corresponding system and observation matrices in both nominal and degraded systems, the cross-variance (CV) function between the nominal SS and the degraded SS, the auto-variance (AV) function of the degraded SS, and the variance of the WGON.

This paper is organized as follows. Section 2 introduces the nominal and degraded state-space models, as well as the realization of the degraded system. Section 3 discusses the robust prediction problem and proposes a Volterra-type integral equation to solve it. In Section 4, Theorem 1 presents the robust RLS prediction and filtering algorithm using the covariance information. In Section 5, Theorem 2 presents the robust RLS Wiener prediction and filtering algorithm. Section 6 presents a recursive algorithm for the prediction error

variance function of the robust RLS predictor described in Theorem 1. Section 7 demonstrates a numerical simulation example for the robust RLS predictor in Theorem 1. The simulation results show the effectiveness of the robust RLS prediction and filtering algorithm.

2 Nominal and Degraded State-Space Models and Realization of Degraded System

Let (1) represent a nominal state-space model in linear continuous-time stochastic systems.

$$\begin{aligned} y(t) &= z(t) + v(t), z(t) = Hx(t), \\ \frac{dx(t)}{dt} &= \Phi x(t) + \Gamma w(t), x(0) = c, \\ E[v(t)v(s)] &= R\delta(t-s), \\ E[w(t)w^T(s)] &= Q\delta(t-s), \\ E[v(t)w^T(s)] &= 0, E[x(0)w^T(t)] = 0, \\ E[x(0)v(t)] &= 0, 0 \leq s, t \end{aligned} \quad (1)$$

Here, $x(t) \in R^n$ is the state vector, and $z(t) \in R^m$ is the signal vector. The input noise $w(t) \in R^l$ and the observation noise $v(t)$ are independent white Gaussian noises with zero mean. Γ is the $n \times l$ input matrix, and H is the $m \times n$ observation matrix. The covariance functions for the input noise $w(t)$ and the observation noise $v(t)$ are given in (1), respectively. (2) presents the state and observation models containing uncertain parameters.

$$\begin{aligned} \tilde{y}(t) &= \tilde{z}(t) + v(t), \\ \tilde{z}(t) &= \tilde{H}(t)\tilde{x}(t), \tilde{H}(t) = H + \Delta H(t), \\ \frac{d\tilde{x}(t)}{dt} &= \tilde{\Phi}(t)\tilde{x}(t) + \Gamma w(t), \tilde{x}(0) = \tilde{c}, \\ \tilde{\Phi}(t) &= \Phi + \Delta\Phi(t), \\ E[\Delta A(t)w^T(s)] &= 0, \\ E[\Delta H(t)v(s)] &= 0, E[\tilde{x}(0)w^T(t)] = 0, \\ E[\tilde{x}(0)v(t)] &= 0, 0 \leq s, t \end{aligned} \quad (2)$$

In (2), $\tilde{\Phi}(t)$ and $\tilde{H}(t)$ denote the degraded system and observation matrices, respectively. The matrices $\Delta A(t)$ and $\Delta H(t)$ represent the corresponding uncertainties. The initial SS $\tilde{x}(0)$ is a random vector that is uncorrelated with both the system input noise $w(t)$ and the measurement noise $v(t)$.

Assume that the degraded signal is represented as $\tilde{z}(t) = \tilde{H}\tilde{x}(t)$, where $\tilde{x}(t)$ is the degraded state vector consisting of n components.

$$\begin{aligned} \tilde{z}(t) &= \tilde{H}\tilde{x}(t), \tilde{z}(t) = \tilde{x}_1(t), \\ \tilde{H} &= [I_{m \times m} \quad 0 \quad 0 \quad \cdots \quad 0], \\ \tilde{x}(t) &= \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \vdots \\ \tilde{x}_{n-1}(t) \\ \tilde{x}_n(t) \end{bmatrix} \end{aligned} \quad (3)$$

Let $\tilde{x}_1(t)$ satisfy a differential equation

$$\begin{aligned} \frac{d\tilde{x}_1^n(t)}{dt^n} &= -\tilde{a}_1 \frac{d\tilde{x}_1^{n-1}(t)}{dt^{n-1}} - \tilde{a}_2 \frac{d\tilde{x}_1^{n-2}(t)}{dt^{n-2}} \\ &\cdots - \tilde{a}_{n-1} \frac{d\tilde{x}_1(t)}{dt} - \tilde{a}_n \tilde{x}_1(t) + \xi(t). \end{aligned} \quad (4)$$

(4) is transformed into a set of state differential equations expressed in the observable companion form of (5).

$$\begin{aligned} \frac{d\tilde{x}(t)}{dt} &= \tilde{\Phi}\tilde{x}(t) + \tilde{\Gamma}\xi(t), \\ E[\xi(t)\xi^T(s)] &= \tilde{Q}\delta(t-s), \\ \tilde{\Phi} &= \begin{bmatrix} 0 & I_{m \times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m \times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m \times m} \\ -\tilde{a}_n & -\tilde{a}_{n-1} & -\tilde{a}_{n-2} & \cdots & -\tilde{a}_1 \end{bmatrix}, \\ \tilde{\Gamma} &= [0 \quad 0 \quad \cdots \quad 0 \quad I_{l \times m}]^T. \end{aligned} \quad (5)$$

$\xi(t)$ represents the residual arising from the approximation of the degraded signal $\tilde{z}(t)$. The degraded system matrix $\tilde{\Phi}$ is estimated using (6) as described in [5].

$$\begin{aligned} \tilde{\Phi} &= E \left[\frac{d\tilde{x}(t)}{dt} \tilde{x}^T(s) \right] E[\tilde{x}(s)\tilde{x}^T(s)]^{-1}, 0 \leq s < t, \\ \tilde{z}(t) &= \tilde{H}(t)\tilde{x}(t), \tilde{H}(t) = H + \Delta H(t), \\ \tilde{x}^T(t) &= [\tilde{x}_1(t) \quad \tilde{x}_2(t) \quad \cdots \quad \tilde{x}_{n-1}(t) \quad \tilde{x}_n(t)], \tilde{x}_1(t) \\ &= \tilde{z}(t), \tilde{x}_2(t) = \frac{d\tilde{z}(t)}{dt}, \cdots, \\ \tilde{x}_{n-1}(t) &= \frac{d^{n-2}\tilde{z}(t)}{dt^{n-2}}, \tilde{x}_n(t) = \frac{d^{n-1}\tilde{z}(t)}{dt^{n-1}} \end{aligned} \quad (6)$$

Also, $\tilde{H}$ is estimated by

$$\tilde{H} = E[\tilde{y}(t)\tilde{x}^T(t)]E[\tilde{x}(t)\tilde{x}^T(t)]^{-1} \quad (7)$$

[5].

3 Robust Prediction Problem

Let the D-time-ahead prediction estimate, $\hat{x}(t+D, t)$, of $x(t+D)$, be given by formula (8), which is a linear transformation of the degraded observation process, $\tilde{y}(s)$, for $0 \leq s \leq t$.

$$\hat{x}(t+D, t) = \int_0^t h(t, s)\tilde{y}(s)ds \quad (8)$$

Let us consider minimizing the mean-square value J of the prediction error $x(t+D) - \hat{x}(t+D, t)$.

$$J = E[(x(t+D) - \hat{x}(t+D, t))^T \times (x(t+D) - \hat{x}(t+D, t))] \quad (9)$$

According to the orthogonal projection theorem, the prediction estimate $\hat{x}(t+D, t)$, which minimizes the cost function J , satisfies the relation (10).

$$x(t+D) - \hat{x}(t+D, t) \perp \tilde{y}(s) \quad (10)$$

Here, " $\perp$ " denotes orthogonality. (11) follows on from (8) and (11).

$$x(t+D) - \int_0^t h(t, s') \tilde{y}(s') ds' \perp \tilde{y}(s), \quad (11)$$

$$0 \leq s \leq t$$

The optimal impulse response function $h(t, s)$ satisfies (12) according to (11).

$$E[x(t+D) \tilde{y}^T(s)] = \int_0^t h(t, s') E[\tilde{y}(s') \tilde{y}^T(s)] ds', \quad (12)$$

From (1) and (2), (12) reduces to (13).

$$h(t, s) R = E[x(t+D) \tilde{y}^T(s)]$$

$$- \int_0^t h(t, s') \tilde{H} E[\tilde{x}(s') \tilde{x}^T(s)] ds' \tilde{H}^T, \quad (13)$$

$$0 \leq s \leq t$$

Let $K_{x\tilde{y}}(t, s)$ be the CC function between the nominal SS $x(t)$ and the degraded observed value $\tilde{y}(s)$, $K_{x\tilde{x}}(t, s)$ be the CC function between the nominal SS $x(t)$ and the degraded SS $\tilde{x}(s)$, and $\tilde{K}(t, s)$ be the AC function of the degraded SS $\tilde{x}(t)$. Then, by noting that $K_{x\tilde{y}}(t, s) = K_{x\tilde{x}}(t, s) \tilde{H}^T$, we rewrite (13) as (14).

$$h(t, s) R = K_{x\tilde{x}}(t+D, s) \tilde{H}^T$$

$$- \int_0^t h(t, s') \tilde{H} \tilde{K}(s', s) ds' \tilde{H}^T, \quad (14)$$

$$0 \leq s \leq t$$

Express the CC function $K_{x\tilde{x}}(t, s)$ in the semi-degenerate kernel form given by (15) [5].

$$K_{x\tilde{x}}(t, s) = \begin{cases} \alpha(t) \beta^T(s), & 0 \leq s \leq t, \\ \gamma(t) \delta^T(s), & 0 \leq t \leq s, \end{cases} \quad (15)$$

$$\alpha(t) = e^{\Phi t}, \beta^T(s) = e^{-\Phi^T s} K_{x\tilde{x}}(s, s),$$

$$\gamma(t) = K_{x\tilde{x}}(t, t) e^{-\Phi^T t}, \delta^T(s) = e^{\Phi^T s}$$

Similarly, express the AC function $\tilde{K}(t, s)$ of the degraded SS $\tilde{x}(t)$ in the semi-degenerate kernel form given by (16) [5].

$$\tilde{K}(t, s) = \begin{cases} \tilde{A}(t) \tilde{B}^T(s), & 0 \leq s \leq t, \\ \tilde{B}(t) \tilde{A}^T(s), & 0 \leq t \leq s, \end{cases} \quad (16)$$

$$\tilde{A}(t) = e^{\tilde{\Phi} t}, \tilde{B}^T(s) = e^{-\tilde{\Phi}^T s} \tilde{K}(s, s)$$

The objective of the present paper is to design the robust RLS prediction algorithm under the presumptions stated in Sections 2 and 3. Starting with (14), the prediction algorithm for the prediction estimate $\hat{x}(t+D, t)$ of the nominal SS $x(t+D)$ is derived.

4 Robust RLS Prediction and Filtering Algorithm Using Covariance Information

Theorem 1 presents the robust RLS prediction and filtering algorithm for both the signal and the nominal SS. The algorithm utilizes the information of the CC function $K_{x\tilde{x}}(t, s)$ from (15) and the AC function $\tilde{K}(t, s)$ from (16), both expressed in their semi-degenerate kernel forms. Additionally, the algorithm utilizes the observation matrix H , the degraded observation matrix $\tilde{H}$, and the variance R of WGON.

Theorem 1 Let the state-space model for the nominal SS $x(t)$ and the observed value $y(t)$ be given by (1). Let the state-space model for the degraded SS $\tilde{x}(t)$ and the degraded observed value $\tilde{y}(t)$ be given by (2). Let the degraded signal be given by $\tilde{z}(t) = \tilde{H} \tilde{x}(t)$ in (3) and the state differential equations for $\tilde{x}(t)$ be given by (5). Let the CC function $K_{x\tilde{x}}(t, s)$ of the nominal SS $x(t)$ with the degraded SS $\tilde{x}(s)$ be given by (15) in the semi-degenerate kernel form. Let the AC function $\tilde{K}(t, s)$ of the degraded SS $\tilde{x}(t)$ be given by (16) in the semi-degenerate kernel form. Thus, the robust RLS prediction and filtering algorithm for the signal and the nominal SS consists of (17)-(26).

Filtering estimate of the signal $z(t)$:

$$\hat{z}(t, t)$$

$$\hat{z}(t, t) = H \hat{x}(t, t) \quad (17)$$

Filtering estimate of the nominal SS $x(t)$: $\hat{x}(t, t)$

$$\hat{x}(t, t) = \alpha(t) e(t) \quad (18)$$

Prediction estimate of the signal $z(t+D)$: $\hat{z}(t+D, t)$

$$\hat{z}(t+D, t) = H \hat{x}(t+D, t) \quad (19)$$

Prediction estimate of the nominal SS $x(t+D)$: $\hat{x}(t+D, t)$

$$\hat{x}(t+D, t) = \alpha(t+D)e(t) \quad (20)$$

Differential equation for $e(t)$:

$$\frac{de(t)}{dt} = J(t, t)(\tilde{y}(t) - \tilde{H}\tilde{A}(t)e_1(t)) \quad (21)$$

Differential equation for $e_1(t)$:

$$\frac{de_1(t)}{dt} = J_1(t, t)(\tilde{y}(t) - \tilde{H}\tilde{A}(t)e_1(t)) \quad (22)$$

Equation for $J(t, t)$:

$$J(t, t) = (\beta^T(t)\tilde{H}^T - r(t)\tilde{A}^T(t)\tilde{H}^T)R^{-1} \quad (23)$$

Equation for $J_1(t, t)$:

$$J_1(t, t) = (\tilde{B}^T(t)\tilde{H}^T - r_1(t)\tilde{A}^T(t)\tilde{H}^T)R^{-1} \quad (24)$$

Differential equation for $r(t)$:

$$\frac{dr(t)}{dt} = J(t, t)(\tilde{H}\tilde{B}(t) - \tilde{H}\tilde{A}(t)r_1(t)), \quad r(0)=0 \quad (25)$$

Differential equation for $r_1(t)$:

$$\frac{dr_1(t)}{dt} = J_1(t, t)(\tilde{H}\tilde{B}(t) - \tilde{H}\tilde{A}(t)r_1(t)), \quad r_1(0)=0 \quad (26)$$

For the prediction and filtering algorithm in Theorem 1 to be asymptotically stable, the real parts of the eigenvalues of $-J_1(t, t)\tilde{H}\tilde{A}(t)$ must be negative.

The proof of Theorem 1 is given in Appendix A.

5 Robust RLS Wiener Prediction and Filtering Algorithm

Theorem 2 presents the robust RLS Wiener prediction and filtering algorithm for the signal and the nominal SS. The algorithm utilizes the system matrices Φ and $\tilde{\Phi}$ for the nominal and degraded systems, the CV function $K_{x\tilde{x}}(t, t)$ between the nominal SS and the degraded SS, the AV function $\tilde{K}(t, t)$ of the degraded SS, the observation matrices H and $\tilde{H}$ for the nominal and degraded systems, and the variance R of the WGON.

Theorem 2 Let the state-space model for the nominal SS $x(t)$ and the observed value $y(t)$ be given by (1). Let the state-space model for the degraded SS $\tilde{x}(t)$ and the degraded observed value $\tilde{y}(t)$ be given by (2). Let the degraded signal be given by $\tilde{z}(t) = \tilde{H}\tilde{x}(t)$ in (3) and the state differential equations for $\tilde{x}(t)$ be given by (5). Then, the robust RLS Wiener prediction and filtering algorithm for the signal and the nominal SS consists

of (27)-(33). The algorithm uses the system matrices Φ and $\tilde{\Phi}$ for the nominal and degraded systems, the CV function $K_{x\tilde{x}}(t, t)$ between the nominal SS $x(t)$ and the degraded SS $\tilde{x}(t)$, the AV function $\tilde{K}(t, t)$ of $\tilde{x}(t)$, the observation matrices H and $\tilde{H}$ for the nominal and degraded systems, and the variance R of the WGON.

Filtering estimate of the signal $z(t)$: $\hat{z}(t, t)$

$$\hat{z}(t, t) = H\hat{x}(t, t) \quad (27)$$

Prediction estimate of the signal $z(t+D)$: $\hat{z}(t+D, t)$

$$\hat{z}(t+D, t) = H\hat{x}(t+D, t) \quad (28)$$

Differential equation for the prediction estimate $\hat{x}(t+D, t)$ of the nominal SS $x(t+D)$:

$$\begin{aligned} \frac{d\hat{x}(t+D, t)}{dt} &= e^{\Phi D}\Phi\hat{x}(t, t) \\ &+ e^{\Phi D}(K_{x\tilde{x}}(t, t)\tilde{H}^T - S(t)\tilde{H}^T) \\ &\times R^{-1}(\tilde{y}(t) - \tilde{H}\hat{x}(t, t)), \end{aligned} \quad (29)$$

Differential equation for the filtering estimate $\hat{x}(t, t)$ of the nominal SS $x(t)$:

$$\begin{aligned} \frac{d\hat{x}(t, t)}{dt} &= \Phi\hat{x}(t, t) \\ &+ (K_{x\tilde{x}}(t, t)\tilde{H}^T - S(t)\tilde{H}^T) \\ &\times R^{-1}(\tilde{y}(t) - \tilde{H}\hat{x}(t, t)), \end{aligned} \quad (30)$$

Differential equation for the filtering estimate $\hat{\tilde{x}}(t, t)$ of the degraded SS $\tilde{x}(t)$:

$$\begin{aligned} \frac{d\hat{\tilde{x}}(t, t)}{dt} &= \tilde{\Phi}\hat{\tilde{x}}(t, t) \\ &+ (\tilde{K}(t, t)\tilde{H}^T - \tilde{S}(t)\tilde{H}^T) \\ &\times R^{-1}(\tilde{y}(t) - \tilde{H}\hat{\tilde{x}}(t, t)), \end{aligned} \quad (31)$$

Differential equation for the function $S(t)$:

$$\begin{aligned} \frac{dS(t)}{dt} &= \Phi S(t) + S(t)\tilde{\Phi}^T \\ &+ (K_{x\tilde{x}}(t, t)\tilde{H}^T - S(t)\tilde{H}^T) \\ &\times R^{-1}(\tilde{H}\tilde{K}(t, t) - \tilde{H}\tilde{S}(t)), \end{aligned} \quad (32)$$

$$S(0) = 0$$

Differential equation for the function $\tilde{S}(t)$:

$$\begin{aligned} \frac{d\tilde{S}(t)}{dt} &= \tilde{\Phi}\tilde{S}(t) + \tilde{S}(t)\tilde{\Phi}^T \\ &+ (\tilde{K}(t, t)\tilde{H}^T - \tilde{S}(t)\tilde{H}^T) \\ &\times R^{-1}(\tilde{H}\tilde{K}(t, t) - \tilde{H}\tilde{S}(t)), \end{aligned} \quad (33)$$

$$\tilde{S}(0) = 0$$

For the prediction and filtering algorithm in Theorem 2 to be stable, the real parts of the

eigenvalues of $-\Phi(\tilde{K}(t, t)\tilde{H}^T - \tilde{S}(t)\tilde{H}^T)R^{-1}\tilde{H}$ must be negative.

The proof of Theorem 2 is given in Appendix B.

6 Prediction Error Variance Function

This section presents an algorithm for calculating the prediction error variance function. The prediction error variance function, $P(t)$, is given by (34).

$$P(t) = E[(x(t+D) - \hat{x}(t+D, t)) \times (x(t+D) - \hat{x}(t+D, t))^T] \quad (34)$$

Since $x(t+D) - \hat{x}(t+D, t)$ is orthogonal to $\hat{x}(t+D, t)$, (34) is transformed into (35).

$$P(t) = E[x(t+D)x(t+D)^T] - E[\hat{x}(t+D, t)x^T(t+D)] \quad (35)$$

Denoting $K_x(t+D, t+D) = E[x(t+D)x(t+D)^T]$ and substituting (8) into (35) lead to (36).

$$P(t) = K_x(t+D, t+D) - \int_0^t h(t, s)E[\tilde{y}(s)x^T(t+D)]ds \quad (36)$$

From (2), (3), and (15), (36) is transformed into (37).

$$\begin{aligned} P(t) &= K_x(t+D, t+D) \\ &- \int_0^t h(t, s)E[x(t+D)\tilde{x}^T(s)]^T ds \tilde{H}^T \\ &= K_x(t+D, t+D) \\ &- \int_0^t h(t, s)E[x(t+D)\tilde{x}^T(s)]ds \tilde{H}^T \\ &= K_x(t+D, t+D) \\ &- \int_0^t h(t, s)\beta(s)ds\alpha^T(t+D)\tilde{H}^T \\ &= K_x(t+D, t+D) \\ &- \int_0^t h(t, s)\beta(s)ds\alpha^T(t)e^{\Phi^T D}\tilde{H}^T \end{aligned} \quad (37)$$

Substituting (A-2) into (37) and introducing the function

$$q(t) = \int_0^t J(t, s)\beta(s)ds, \quad (38)$$

(37) is transformed into (39).

$$P(t) = K_x(t+D, t+D) - q(t)\alpha^T(t)e^{\Phi^T D}\tilde{H}^T \quad (39)$$

Taking the derivative of (38) with respect to t gives (40).

$$\frac{dq(t)}{dt} = J(t, t)\beta(t) + \int_0^t \frac{\partial J(t, s)}{\partial t}\beta(s)ds \quad (40)$$

Substituting (A-5) into (40) and introducing the function

$$q_1(t) = \int_0^t J_1(t, s)\beta(s)ds, \quad (41)$$

(40) is transformed into (42).

$$\begin{aligned} \frac{dq_1(t)}{dt} &= J(t, t)(\beta(t) - \tilde{H}\tilde{A}(t)q_1(t)), \\ q_1(0) &= 0 \end{aligned} \quad (42)$$

Differentiating (41) with respect to t yields (43).

$$\begin{aligned} \frac{dq_1(t)}{dt} &= J_1(t, t)\beta(t) \\ &+ \int_0^t \frac{\partial J_1(t, s)}{\partial t}\beta(s)ds \end{aligned} \quad (43)$$

Substituting (A-13) into (43) results in (44).

$$\begin{aligned} \frac{dq_1(t)}{dt} &= J_1(t, t)(\beta(t) - \tilde{H}\tilde{A}(t)q_1(t)), \\ q_1(0) &= 0 \end{aligned} \quad (44)$$

The recursive algorithm for the prediction error variance function, $P(t)$, consists of equations (24), (25), (26), (27), (39), (42), and (44).

7 A Numerical Simulation Example

Let the observation equation for the signal $z(t)$ and the state differential equations for $x(t)$ be given by (45).

$$\begin{aligned} y(t) &= z(t) + v(t), z(t) = Hx(t), \\ H &= [1 \ 0], \\ \frac{dx(t)}{dt} &= \Phi x(t) + \Gamma w(t), x(0) = c, \\ x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \Phi &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \omega_n = \sqrt{3}, \\ \zeta &= \frac{2}{\omega_n}, \Gamma = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}, \\ E[v(k)v(s)] &= R\delta(t-s), R > 0, \\ E[w(t)w(s)] &= Q\delta(t-s), Q = 1, \\ E[v(t)w(s)] &= 0, E[x(0)w(t)] = 0, \\ E[x(0)v(t)] &= 0 \end{aligned} \quad (45)$$

Let the observation equation for the degraded signal $\tilde{z}(t)$, and the state differential equations for the degraded SS $\tilde{x}(t)$ be given by (46).

$$\begin{aligned}
 \tilde{y}(t) &= \tilde{z}(t) + v(t), \\
 \tilde{z}(t) &= \tilde{H}(t)\tilde{x}(t), \tilde{H}(t) = H + \Delta H(t), \\
 \Delta H(t) &= [0.1 \ 0], \\
 \frac{d\tilde{x}(t)}{dt} &= \tilde{\Phi}(t)\tilde{x}(t) + \Gamma w(t), \\
 \tilde{\Phi}(t) &= \Phi + \Delta\Phi(t), \\
 \Delta\Phi(t) &= \begin{bmatrix} 0 & 0 \\ -0.1 * rand & -0.1 * rand \end{bmatrix}, \\
 E[v(t)w(s)] &= 0, E[\Delta\Phi(t)w(s)] = 0, \\
 E[\Delta C(t)v(s)] &= 0, E[\tilde{x}(0)w(t)] = 0, \\
 E[\tilde{x}(0)v(t)] &= 0
 \end{aligned} \tag{46}$$

$\Delta\Phi(t)$ represents an uncertain matrix that is additional to the system matrix Φ . "rand" represents a scalar random number from a uniform distribution in the interval (0, 1).

By substituting $\alpha(t+D)$ and $\beta(t)$ in the CC function $K_{x\tilde{x}}(t+D, t)$ of the nominal SS $x(t+D)$ with the degraded SS $\tilde{x}(t)$, and $\tilde{A}(t)$ and $\tilde{B}(t)$ in the AV function $\tilde{K}(t, t)$ of $\tilde{x}(t)$, observation matrices H and $\tilde{H}$, and the variance R of WGON into the robust prediction and filtering algorithm of Theorem 1, the prediction and filtering estimates are computed recursively. Figure 1 illustrates the signal $z(t)$ and its prediction estimate $\hat{z}(t+D, t)$, $D = 0.005$, vs. t for the WGON $N(0, 0.3^2)$. From Figure 1, the prediction estimate $\hat{z}(t+D, t)$ gradually approaches the signal $z(t+D)$ as time t progresses. Figure 2 illustrates the signal $z(t)$ and its prediction estimate $\hat{z}(t+D, t)$, $D = 0.005$, vs. t for the WGON $N(0, 0.3^2)$ by the RLS predictor proposed in [20]. In Figure 2, the prediction estimate $\hat{z}(t+0.005, t)$ gradually

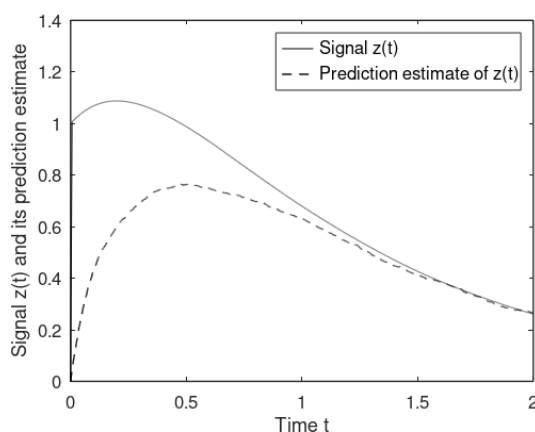


Fig. 1: Signal $z(t)$ and its prediction estimate $z(t+0.005, t)$ vs. t for the WGON $N(0, 0.3^2)$ by Theorem 1

Source: created by the author.

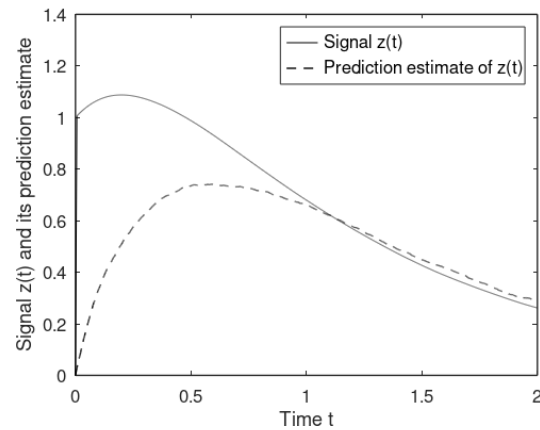


Fig. 2: Signal $z(t)$ and its prediction estimate $z(t+0.005, t)$ vs. t for the WGON $N(0, 0.3^2)$ by the RLS prediction algorithm in [20]

Source: created by the author.

approaches the signal $z(t+0.005)$ as time t progresses. Figure 3 illustrates the mean-square values (MSVs) of prediction errors $z(t+D) - \hat{z}(t+D, t)$, $0 \leq t \leq 2$, $0 \leq D \leq 0.005$, by the robust RLS filtering and prediction algorithm of Theorem 1 for the white Gaussian observation noises (WGONs) $N(0, 0.1^2)$, $N(0, 0.3^2)$ and $N(0, 0.5^2)$. Here, the MSVs of the filtering and prediction errors are evaluated by $\frac{1}{2000} \sum_{i=1}^{2000} (z(i\Delta) - \hat{z}(i\Delta, i\Delta))^2$ and $\frac{1}{2000} \sum_{i=1}^{2000} (z(i\Delta + D) - \hat{z}(i\Delta + D, i\Delta))^2$, $\Delta = 0.001$, $0 \leq D \leq 0.005$, respectively. Except for the observation noise $N(0, 0.5^2)$, the MSV shows almost the same value even if the fixed lag D is varied. As the variance of the observation noise increases, the MSV of the prediction errors grows for every D-step-ahead prediction. Figure 4

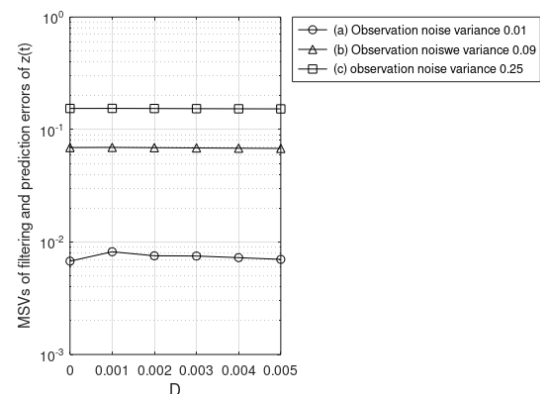


Fig. 3: MSVs of filtering errors $z_t - \hat{z}_t, t$ and prediction errors $z_{t+D} - \hat{z}_{t+D, t}$, $0 \leq t \leq 2$, $\Delta = 0.001$, $0 \leq D \leq 0.005$, by the robust RLS filtering and prediction algorithm in Theorem 1 for the WGONs $N(0, 0.1^2)$, $N(0, 0.3^2)$ and $N(0, 0.5^2)$

Source: created by the author.

illustrates the MSVs of prediction errors $z(t + D) - \hat{z}(t + D, t)$ and prediction errors $z(t + D) - \hat{z}(t + D, t)$, $0 \leq t \leq 2, 0 \leq D \leq 0.005$, by the RLS filtering and prediction algorithm in [20] for the WGNs $N(0, 0.1^2)$, $N(0, 0.3^2)$ and $N(0, 0.5^2)$. From Figure 3 and Figure 4, the filtering and prediction accuracies obtained by the algorithm in Theorem 1 are superior to those of the RLS filter and predictor presented in [20].

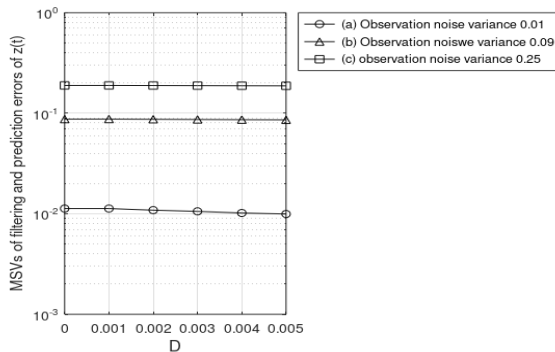


Fig. 4: MSVs of filtering errors $z_t - \hat{z}_{t,t}$ and prediction errors $z_{t+D} - \hat{z}_{t+D,t}$, $0 \leq t \leq 2$, $\Delta = 0.001$, $0 \leq D \leq 0.005$, by the RLS filtering and prediction algorithm [20] for the WGNs $N(0, 0.12)$, $N(0, 0.32)$ and $N(0, 0.52)$

Source: created by the author.

8 Conclusion

This paper, in Theorem 1, presented the robust RLS prediction and filtering algorithm for the signal $z(t)$ and the nominal SS $x(t)$. The algorithm utilizes the information of the CC function $K_{xx}(t, s)$ from (15) and the AC function $\tilde{K}(t, s)$ from (16), both expressed in their semi-degenerate kernel forms. Additionally, the algorithm uses the observation matrix H , the degraded observation matrix $\tilde{H}$, and the variance R of WGN. The robust RLS Wiener prediction and filtering algorithm for the signal and the nominal SS is presented in Theorem 2. It utilizes the system matrix Φ and the degraded system matrix $\tilde{\Phi}$, as well as the observation matrix H for the nominal system and the observation matrix $\tilde{H}$ for the degraded system. It also uses the CV function $K_{xx}(t, t)$ and the AV function $\tilde{K}(t, t)$, as well as the variance R of the WGN.

A numerical simulation is presented to demonstrate the effectiveness of the robust RLS prediction and filtering algorithm presented in Theorem 1. As time t progresses, the prediction estimate $\hat{z}(t + 0.005, t)$ in Figure 1 gradually approaches the signal $(t + 0.005)$. The numerical simulation results show that the robust RLS

prediction and filtering algorithm presented in Theorem 1 is effective. Figure 3 and Figure 4 show that the filtering and prediction accuracies achieved by the algorithm in Theorem 1 for the signal are superior to those obtained by the RLS filter and predictor presented in [20].

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author(s) utilized artificial intelligence (AI) technologies solely for the purpose of linguistic refinement and grammatical correction. Following the application of these tools, the author(s) comprehensively reviewed, verified, and edited the resulting text. The author(s) maintain full accountability and responsibility for the accuracy, integrity, and final content of the publication.

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Conflict of Interest

The authors have no conflicts of interest to declare.

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APPENDIX A

Introducing the integral equation

$$J(t, s)R = \beta^T(s)\check{H}^T - \int_0^t J(t, s')\check{H}\check{K}(s', s)ds'\check{H}^T, \quad (\text{A-1})$$

from (14), $h(t, s)$ is given by (A-2).

$$h(t, s) = \alpha(t + D)J(t, s) \quad (\text{A-2})$$

Differentiating (A-1) with respect to t , we have (A-3).

$$\begin{aligned} \frac{\partial J(t, s)}{\partial t} R &= -J(t, t)\check{H}\check{K}(t, s)\check{H}^T \\ &\quad - \int_0^t \frac{\partial J(t, s')}{\partial t} \check{H}\check{K}(s', s)ds'\check{H}^T \end{aligned} \quad (\text{A-3})$$

Introducing

$$\begin{aligned} J_1(t, s)R &= \check{B}^T(s)\check{H}^T \\ &\quad - \int_0^t J_1(t, s')\check{H}\check{K}(s', s)ds'\check{H}^T, \end{aligned} \quad (\text{A-4})$$

$\frac{\partial J(t, s)}{\partial t}$ is given by (A-5).

$$\frac{\partial J(t, s)}{\partial t} = -J(t, t)\check{H}\check{A}(t)J_1(t, s) \quad (\text{A-5})$$

Introducing

$$r(t) = \int_0^t J(t, s')\check{H}\check{B}(s')ds', \quad (\text{A-6})$$

from (A-1), $J(t, t)$ satisfies (A-7).

$$\begin{aligned} J(t, t)R &= \beta^T(t)\check{H}^T \\ &\quad - \int_0^t J(t, s')\check{H}\check{K}(s', t)ds'\check{H}^T \\ &= \beta^T(t)\check{H}^T - r(t)\check{A}^T(t)\check{H}^T \end{aligned} \quad (\text{A-7})$$

Differentiating $r(t)$ with respect to t , from (A-5), we have (A-8).

$$\begin{aligned} \frac{dr(t)}{dt} &= J(t, t)\check{H}\check{B}(t) \\ &\quad + \int_0^t \frac{\partial J(t, s')}{\partial t} \check{H}\check{B}(s')ds' \\ &= J(t, t)\check{H}\check{B}(t) \\ &\quad - J(t, t)\check{H}\check{A}(t) \int_0^t J_1(t, s')\check{H}\check{B}(s')ds' \end{aligned} \quad (\text{A-8})$$

Introducing the function

$$r_1(t) = \int_0^t J_1(t, s')\check{H}\check{B}(s')ds', \quad (\text{A-9})$$

we obtain (A-10).

$$\begin{aligned} \frac{dr(t)}{dt} &= J(t, t)(\check{H}\check{B}(t) - \check{H}\check{A}(t)r_1(t)), \\ r(0) &= 0 \end{aligned} \quad (\text{A-10})$$

From (A-4), (A-9), and (16), $J_1(t, t)$ satisfies (A-11).

$$\begin{aligned}
 J_1(t, t)R &= \check{B}^T(t)\check{H}^T \\
 &- \int_0^t J_1(t, s')\check{H}\check{K}(s', t)ds'\check{H}^T \\
 &= \check{B}^T(t)\check{H}^T \\
 &- \int_0^t J_1(t, s')\check{H}\check{B}(s')ds'\check{A}^T(t)\check{H}^T \\
 &= \check{B}^T(t)\check{H}^T - r_1(t)\check{A}^T(t)\check{H}^T
 \end{aligned} \tag{A-11}$$

Differentiating (A-4) with respect to t , we have (A-12).

$$\begin{aligned}
 \frac{\partial J_1(t, s)}{\partial t}R &= -J_1(t, t)\check{H}\check{K}(t, s)\check{H}^T \\
 &- \int_0^t \frac{\partial J_1(t, s')}{\partial t}\check{H}\check{K}(s', s)ds'\check{H}^T \\
 &= -J_1(t, t)\check{H}\check{K}(t, s)\check{H}^T \\
 &- \int_0^t \frac{\partial J_1(t, s')}{\partial t}\check{H}\check{K}(s', s)ds'\check{H}^T = \check{B}^T(t)\check{H}^T - r_1(t)\check{A}^T(t)\check{H}^T
 \end{aligned} \tag{A-12}$$

From (A-4), $\frac{\partial J_1(t, s)}{\partial t}$ is given by (A-13).

$$\frac{\partial J_1(t, s)}{\partial t} = -J_1(t, t)\check{H}\check{A}(t)J_1(t, s) \tag{A-13}$$

Differentiating (A-9) with respect to t , we obtain (A-14).

$$\begin{aligned}
 \frac{dr_1(t)}{dt} &= J_1(t, t)\check{H}\check{B}(t) \\
 &+ \int_0^t \frac{\partial J_1(t, s')}{\partial t}\check{H}\check{B}(s')ds' \\
 &= J_1(t, t)\check{H}\check{B}(t) \\
 &- J_1(t, t)\check{H}\check{A}(t) \int_0^t J_1(t, s')\check{H}\check{B}(s')ds' \\
 &= J_1(t, t)(\check{H}\check{B}(t) - \check{H}\check{A}(t)r_1(t)), \\
 r_1(0) &= 0
 \end{aligned} \tag{A-14}$$

The prediction estimate $\hat{x}(t + D, t)$ of $x(t + D)$ is given by (8). Substituting (A-2) into (8), we have (A-15).

$$\begin{aligned}
 \hat{x}(t + D, t) &= \alpha(t + D) \\
 &\times \int_0^t J(t, s')\check{y}(s')ds'
 \end{aligned} \tag{A-15}$$

Introducing

$$e(t) = \int_0^t J(t, s')\check{y}(s')ds', \tag{A-16}$$

we obtain

$$\hat{x}(t + D, t) = \alpha(t + D)e(t), \tag{A-17}$$

Differentiating (A-16) with respect to t , from (A-5), we obtain (A-18).

$$\begin{aligned}
 \frac{de(t)}{dt} &= J(t, t)\tilde{y}(t) \\
 &- J(t, t)\tilde{H}\tilde{A}(t) \int_0^t J_1(t, s')\tilde{y}(s')ds' \\
 &= J(t, t)(\tilde{y}(t) - \tilde{H}\tilde{A}(t)e_1(t)), \\
 e(0) &= 0
 \end{aligned} \tag{A-18}$$

Here, $e_1(t)$ is given by (A-19).

$$e_1(t) = \int_0^t J_1(t, s')\tilde{y}(s')ds', \tag{A-19}$$

Differentiating (A-19) with respect to t , from (A-13), we obtain (A-20).

$$\begin{aligned}
 \frac{de_1(t)}{dt} &= J_1(t, t)\tilde{y}(t) \\
 &- J_1(t, t)\tilde{H}\tilde{A}(t) \int_0^t J_1(t, s)\tilde{y}(s')ds' \\
 &= J_1(t, t)(\tilde{y}(t) - \tilde{H}\tilde{A}(t)e_1(t)), \\
 e_1(0) &= 0
 \end{aligned} \tag{A-20}$$

(Q.E.D.)

APPENDIX B

The prediction estimate $\hat{x}(t + D, t)$ of the state $x(t + D)$ is given by (20). Differentiating (20) with respect to t , from (15) and (21), we have (B-1).

$$\begin{aligned}
 \frac{d\hat{x}(t + D, t)}{dt} &= e^{\Phi D} \frac{d\alpha(t)}{dt} e(t) \\
 &+ e^{\Phi D} \alpha(t) \frac{de(t)}{dt} \\
 &= e^{\Phi D} \Phi \alpha(t) e(t) + e^{\Phi D} \alpha(t) J(t, t) \\
 &\times (\tilde{y}(t) - \tilde{H}\tilde{A}(t)e_1(t))
 \end{aligned} \tag{B-1}$$

Noting that $\alpha(t)e(t)$ is the filtering estimate $\hat{x}(t, t)$ and substituting (23) into (B-1), we have (B-2).

$$\begin{aligned}
 \frac{d\hat{x}(t + D, t)}{dt} &= e^{\Phi D} \frac{d\alpha(t)}{dt} e(t) \\
 &+ e^{\Phi D} \alpha(t) \\
 &\times (\beta^T(t)\tilde{H}^T - r(t)\tilde{A}^T(t)\tilde{H}^T)R^{-1} \\
 &\times (\tilde{y}(t) - \tilde{H}\tilde{A}(t)e_1(t))
 \end{aligned} \tag{B-2}$$

Putting

$$S(t) = \alpha(t) r(t)\tilde{A}^T(t) \tag{B-3}$$

and noting that $\tilde{A}(t)e_1(t)$ is the filtering estimate $\hat{\tilde{x}}(t, t)$ of the degraded state $\tilde{x}(t)$, that is,

$$\hat{\tilde{x}}(t, t) = \tilde{A}(t)e_1(t) \tag{B-4}$$

from (15), we obtain (B-5).

$$\begin{aligned}
 \frac{d\hat{x}(t + D, t)}{dt} &= e^{\Phi D} \Phi \hat{x}(t, t) \\
 &+ e^{\Phi D} \\
 &\times (K_{x\tilde{x}}(t, t)\tilde{H}^T - S(t)\tilde{H}^T)R^{-1} \\
 &\times (\tilde{y}(t) - \tilde{H}\hat{\tilde{x}}(t, t)), \\
 \hat{x}(D, 0) &= e^{\Phi D} \hat{x}(0, 0) = 0
 \end{aligned} \tag{B-5}$$

$\check{y}(t) - \check{H}\hat{x}(t, t)$ is the innovation process. Differentiating (B-3), with respect to t , from (15), (16), (23), (25), and (B-3), we have (B-6).

$$\begin{aligned} \frac{dS(t)}{dt} &= \frac{d\alpha(t)}{dt} r(t) \check{A}^T(t) \\ &+ \alpha(t) \frac{dr(t)}{dt} \check{A}^T + \alpha(t) r(t) \frac{d\check{A}^T(t)}{dt} \\ &= \Phi\alpha(t) r(t) \check{A}^T(t) \\ &+ \alpha(t) \frac{dr(t)}{dt} \check{A}^T(t) \\ &+ \alpha(t) r(t) \check{A}^T(t) \check{\Phi}^T \\ &= \Phi S(t) + (K_{x\check{x}}(t, t) \check{H}^T - S(t) \check{H}^T) \\ &R^{-1}(\check{H}\check{K}(t, t) - \check{H}\check{A}(t)r_1(t)\check{A}^T(t)) \\ &+ S(t) \check{\Phi}^T \end{aligned} \quad (B-6)$$

Introducing the function

$$\check{S}(t) = \check{A}(t)r_1(t)\check{A}^T(t), \quad (B-7)$$

we obtain (B-8).

$$\begin{aligned} \frac{dS(t)}{dt} &= \Phi S(t) + S(t) \check{\Phi}^T \\ &+ (K_{x\check{x}}(t, t) \check{H}^T - S(t) \check{H}^T) R^{-1} \\ &= \Phi S(t) + (K_{x\check{x}}(t, t) \check{H}^T - S(t) \check{H}^T) \\ &\times R^{-1}(\check{H}\check{K}(t, t) - \check{H}\check{S}(t)), \\ &S(0)=0 \end{aligned} \quad (B-8)$$

Differentiating (B-7) with respect to t , from (16), (24) and (26), we have (B-9).

$$\begin{aligned} \frac{d\check{S}(t)}{dt} &= \frac{d\check{A}(t)}{dt} r_1(t) \check{A}^T(t) \\ &+ \check{A}(t) \frac{dr_1(t)}{dt} \check{A}^T(t) \\ &+ \check{A}(t) r_1(t) \frac{d\check{A}^T(t)}{dt} \\ &= \check{\Phi} \check{A}(t) r_1(t) \check{A}^T(t) \\ &+ \check{A}(t) J_1(t, t) \\ &\times (\check{H}\check{B}(t) - \check{H}\check{A}(t)r_1(t)) \check{A}^T(t) \\ &+ \check{A}(t) r_1(t) \check{A}^T(t) \check{\Phi}^T \\ &= \check{\Phi} \check{S}(t) + \check{S}(t) \check{\Phi}^T \\ &+ (\check{K}(t, t) \check{H}^T - \check{S}(t) \check{H}^T) \\ &\times R^{-1}(\check{H}\check{K}(t, t) - \check{H}\check{S}(t)), \\ &\check{S}(0) = 0 \end{aligned} \quad (B-9)$$

Differentiating (B-3) with respect to t , from (16), (22), (24), and (B-7), we obtain (B-10).

$$\begin{aligned}
 \frac{d\hat{\tilde{x}}(t,t)}{dt} &= \frac{d\check{A}(t)}{dt} e_1(t) + \check{A}(t) \frac{de_1(t)}{dt} \\
 &= \check{\Phi}\check{A}(t)e_1(t) + \check{A}(t)J_1(t,t) \\
 &\times (\check{y}(t) - \check{H}\check{A}(t)e_1(t)) \\
 &= \check{\Phi}\hat{\tilde{x}}(t,t) \\
 &+ \check{A}(t)(\check{B}^T(t)\check{H}^T - r_1(t)\check{A}^T(t)\check{H}^T) \\
 &\times R^{-1}(\check{y}(t) - \check{H}\hat{\tilde{x}}(t,t)) \\
 &= \check{\Phi}\hat{\tilde{x}}(t,t) + (\check{K}(t,t)\check{H}^T - \check{S}(t)\check{H}^T) \\
 &\times R^{-1}(\check{y}(t) - \check{H}\hat{\tilde{x}}(t,t)), \\
 \hat{\tilde{x}}(0,0) &= 0
 \end{aligned}
 \tag{B-10}$$

(Q.E.D.)