Stability Analysis and Efficient Algorithms for Nearly Singular Multidimensional Linear Inverse Problems

ADANE AKATE*, EMŐKE IMRE
Doctoral School of Applied Informatics and Applied Mathematics,
Obuda University, Budapest,
HUNGARY

*Corresponding author

Abstract: - Multidimensional linear inverse problems pop up in various fields of science and engineering, from medical imaging to geophysical exploration. One of the main challenges is their inherent ill-posedness, which often shows up in practice as system matrices that are nearly singular or severely ill-conditioned. In these situations, direct solution methods can become highly sensitive to measurement noise and numerical fluctuations, resulting in unstable and physically meaningless reconstructions. This article delves into the stability analysis of these issues using condition numbers, singular value decomposition, and the Picard condition. It also discusses traditional regularization techniques, particularly Tikhonov regularization. We will take a closer look at how preconditioned iterative solvers can help tackle large-scale problems and provide a detailed MATLAB-based example on 2D image deblurring. This will illustrate how ill-conditioning manifests in real-world scenarios and how regularization and preconditioning work together to stabilize the solution. The interplay between robust regularization and advanced numerical methods is crucial for achieving stable, accurate, and computationally feasible solutions to large-scale, nearly singular inverse problems.

Key-Words: - Linear Inverse Problems, Ill-Posedness, Tikhonov Regularization, Preconditioned Iterative Solvers, Condition Number, Stability Analysis.

Received: March 19, 2026. Revised: May 15, 2026. Accepted: June 13, 2026. Available online: September 16, 2026.

1 Introduction

Linear inverse problems offer a mathematical approach to uncovering unknown quantities from indirect and often noisy measurements. These problems pop up in various fields, including computed tomography (CT) [1], seismic imaging [2], image deblurring [3], and remote sensing [4]. When we look at these issues in a discrete format, they usually take shape as a linear system represented by the equation.

$$Ax = b, \qquad (1)$$

where $A \in \mathbb{R}^{m\times n}$ is the forward operator, $x \in \mathbb{R}^n$ is the unknown vector of parameters, and $b \in \mathbb{R}^m$ is the measured data. Many of these practical problems are considered ill-posed in the Hadamard sense, which means that a solution might not exist, might not be unique, or could be highly sensitive to the input data. In finite-dimensional scenarios, this often translates to the system matrix being poorly conditioned or nearly singular [5]. As a result, even tiny changes in the data vector can cause significant fluctuations in the computed solution, highlighting a strong sensitivity to measurement noise and numerical errors. Recent studies have explored combining Tikhonov regularization with optimization techniques, including genetic algorithms, particularly for inverse heat conduction and other parameter identification challenges. These approaches have shown that with the right regularization, we can achieve stable and accurate solutions even when dealing with noisy data [6], [7]. Other research has focused on the existence, uniqueness, and stability of inverse coefficient problems, proposing regularization algorithms specifically designed for certain PDE models [8], [9]. These findings emphasize that, in realistic scenarios, stability analysis and regularization should go hand in hand with algorithm design. In large-scale multidimensional contexts, like 2D or 3D image deblurring, the challenges become even more pronounced due to the high dimensionality and the significant attenuation of high-frequency components by the forward operator. The singular values of A can drop off quickly toward zero, which results in very high condition numbers [10]. This makes straightforward least squares reconstruction impractical without some form of stabilization. Additionally, directly forming and inverting $A^T A$ is not feasible from a computational standpoint, so we need efficient iterative solvers and matrix-free implementations. Recent studies in image deblurring and other inverse imaging tasks have shown that when you pair appropriate regularization techniques with well-crafted numerical algorithms, you can achieve notable improvements in reconstruction quality. This is often assessed using metrics like PSNR and SSIM [11].

In addition to that, regularization techniques paired with efficient iterative solvers are essential for tackling large linear systems. Mixed-precision preconditioned conjugate gradient (PCG) methods utilize iterative-refinement-based preconditioners and low-precision inner solves to enhance convergence speed while maintaining accuracy. This makes them particularly valuable in scientific computing and imaging applications, where both stability and computational efficiency are crucial. A study by [12] introduced an effective numerical method for addressing systems of linear integro-differential equations, showing that this approach can produce accurate and stable solutions for problems that are tough to handle analytically.

To overcome these challenges, modern strategies blend regularization techniques with efficient iterative solvers for large linear systems. Regularization brings in prior information or constraints to stabilize the inversion of sparse, multidimensional issues. Nowadays, mixed precision preconditioned methods, matrix-free implementations, and problem dependent preconditioners have become standard in large-scale scientific computing.

1.1 Contribution

This study dives into the stability and numerical solutions of nearly singular multidimensional linear inverse problems, focusing on how severe ill-conditioning can impact solution behavior in real-world computations. One of the key contributions is a structured stability analysis that uses singular value decomposition (SVD) and condition number theory. This analysis sheds light on how small singular values can amplify noise and explains why naive least-squares methods often fall short, while also highlighting the crucial stabilizing effect of regularization.

From a computational standpoint, the paper outlines a comprehensive workflow in which Tikhonov regularization is paired with preconditioned iterative solvers to achieve stable and efficient solutions for large-scale problems. A two-dimensional image deblurring example implemented in MATLAB showcases the effects of ill-conditioning, the role of Tikhonov regularization, the impact of preconditioning, and how to choose the regularization parameter using the L curve. Instead of introducing a completely new algorithm, this work combines classical stability theory with a reproducible numerical implementation specifically designed for nearly singular multidimensional problems.

1.2 Novelty

The novelty of this study, which uses SVD, condition numbers, Tikhonov regularization, and preconditioned iterative solvers, is well established; their integrated treatment in the specific context of nearly singular multidimensional inverse problems provides added value. The novelty lies in the focused analysis of how high dimensionality and severe ill-conditioning interact and in a detailed, methodological MATLAB demonstration that directly links singular value decay, regularization strength, iterative convergence, and reconstruction quality.

The paper emphasizes how theoretical stability measures, such as condition numbers and Picard plots, manifest in real computations, bridging the gap between abstract analysis and algorithmic behavior. Furthermore, the studies highlight the importance of preconditioning with regularized formulations to accelerate convergence and a transparent, reproducible computational pipeline that can serve as a practical reference for similar ill-conditioned inverse problems.

The remainder of the paper is organized as follows. Section 2 reviews the mathematical formulation of linear inverse problems, well-posedness and ill-posedness, and stability analysis via condition numbers and singular values. Section 3 outlines regularization strategies, with emphasis on Tikhonov regularization and the choice of the regularization parameter. Section 4 discusses efficient algorithms for large-scale regularized problems, focusing on preconditioned iterative solvers. Section 5 presents a MATLAB-based 2D image deblurring demonstration, including L-curve analysis and qualitative and quantitative analysis of the reconstructions. Section 6: Results and Discussion. Section 7 includes conclusions of the main findings and comments on possible future work.

2 Mathematical Formulations

2.1 Inverse Problems

Inverse problems aim to infer unknown causes or parameters from observed effects. In many applications in medical imaging, geophysics, signal processing, and computer vision [2], [3] the relationship between the unknown vector x and the data y can be expressed as

$$y = Ax + \varepsilon \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$is the forward operator, $x \in \mathbb{R}^{n}$is the parameter vector $y \in \mathbb{R}^{m}$is the observed data, and $\varepsilon \in \mathbb{R}^{m}$ represents measurement noise or modeling error. The objective is to reconstruct a meaningful approximation of x from y.

2.2 Linear Inverse Problems

When the forward map can be approximated as linear, one obtains a linear inverse problem. In the discrete setting, a standard approach is to seek x that minimizes the least-squares functional

$$\min_{x\in\mathbb{R}^n} \|Ax - y\|_2{}^2 \tag{3}$$

If A has full column rank, the minimizer satisfies the normal equations $A^T Ax = A^T y$. However, in inverse problems these normal equations are often ill-conditioned because of rapidly decaying singular values, so the least-squares solution is extremely sensitive to perturbations in y[13].

2.3 Well-Posedness and Ill-Posedness

A problem is well-posed in the sense of Hadamard [13] if its solution exists, is unique, and depends continuously on the data. Linear inverse problems frequently violate one or more of these conditions; in practice, the most critical issue is instability, meaning that small changes in the data cause large changes in the solution [14]. For discrete systems, this is quantified by the condition number, which measures the sensitivity of the solution to perturbations in the data and the operator.

The causes of Ill-Posedness are the following [1]:

1. **Rank Deficiency:** If $rank(A) < n$, the opera tor Ahasanon-trivial null space, meaning multiple distinct x vectors can map to the same Ax. This leads to non-uniqueness.
2. **Incompatibility:** If y does not lie in the range of A, then no exact solution exists, violating existence. This is common with noisy data.
3. **Instability (Near-Singularity):** This is the most prevalent and challenging aspect of ill posed problems. It occurs when the operator A is nearly singular; meaning its inverse (or pseudoinverse) has a very large norm. Small perturbations or noise in the data y are then catastrophically amplified in the computed solution x.

2.4 Stability Analysis

2.3.1 The Condition Number as indicator of ill-posedness

A defining feature of ill-posed linear inverse problems is that the solution does not depend continuously on the data: very small perturbations in the right-hand side can cause very large changes in the solution. In operator-theoretic terms this means that the inverse operator A^{-1} is unbounded. We consider the linear system

$$y = Ax \tag{4}$$

where A is a given matrix and x is the unknown solution. In the ideal noise free case, the formal solution is

$$x = A^{-1}y \tag{5}$$

In practice, we only have noisy data

$$y\delta = y + \delta y, \tag{6}$$

where δy denotes a small perturbation. The corresponding solution is

$$x\delta = A^{-1}y\delta = x + A^{-1}\delta y \tag{7}$$

So the in the reconstructed error satisfies

$$\delta x = x\delta - x = A^{-1}\delta y \tag{8}$$

This expression makes clear that error propagation is governed by the size of A^{-1}: if $\| A^{-1} \|$ is large, modest data noise can be strongly amplified in the solution [15].

2.3.2 Condition number and stability

The sensitivity of the solution to perturbations in both the data and the matrix is quantified by the condition number. For a non-singular matrix A and a given matrix norm $\|\cdot\|$, it is defined as

$$\kappa(A) = \| A \| \| A^{-1} \| \tag{9}$$

For singular or rectangular matrices, the condition number is defined using the Moore–Penrose pseudoinverse A^{+} as

$$\kappa(A) = \| A \| \| A^{+} \| \tag{10}$$

For the linear system $Ax = y$, one can derive the bound

$$\frac{\| \delta x \|}{\| x \|} \leq \kappa(A) \frac{\| \delta y \|}{\| y \|} \tag{11}$$

which shows that a large $\kappa(A)$ allows small relative errors in the data to produce large relative errors in the solution. Matrices with large condition numbers are called ill-conditioned [16]. Equivalently, starting from a perturbed system

$$A(x + \delta x) = b + e \tag{12}$$

and subtracting the original equation (1) gives

$$\delta x = A^{-1}e$$

and, after taking norms and maximizing over all nonzero b and e, one arrives at the same expression for $\kappa(A)$. Thus, the condition number represents the worst-case amplification of relative data errors and depends only on A, not on the numerical method used to solve the system. For any consistent norm,

$$\kappa(A) \geq \| A^{-1}A \| = 1,$$

with equality only when A is (up to a scalar) an isometry. If $\kappa(A)$ is infinite, then A is singular and the problem lacks a unique, stable solution, so some form of regularization is required.

2.4.3 Condition number and Singular values

The condition number of a non-singular matrix A with respect to a consistent matrix norm $\|\cdot\|$ is

$$\kappa(A) = \frac{\| A \|}{\| A^{-1} \|}$$

For rectangular or singular matrices, one uses the pseudoinverse A^{+} instead of A^{-1}. When the norm is induced by the Euclidean norm, the condition number admits the spectral characterization. Let A have a singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > 0$. then

$$\kappa_2(A) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)}$$

If A is normal, the 2-norm condition number reduces to

$$\kappa_2(A) = \frac{max|\lambda(A)|}{min|\lambda(A)|}$$

where σ_{max} and σ_{min} are the largest and smallest nonzero singular values of A and $\lambda(A)$ are the eigenvalues. In the special case where A is unitary, all singular values are equal to one and $\kappa_2(A) = 1$.

Large condition numbers are typical of discretized inverse problems, where the singular values decay rapidly toward zero. Such matrices are nearly singular, and solving the associated linear systems without regularization leads to severe numerical instability [17], [18]. This observation motivates the use of regularization techniques, discussed in the following sections, to restore stability and obtain meaningful solutions.

Perturbation bounds for systems with errors in both matrix and data show that the relative error in the solution can grow proportionally to $\kappa(A)$ times the relative perturbations. This provides a direct link between the singular value spectrum, condition number, and practical numerical stability [16]. In many applications, both the system matrix and the right-hand side are affected by errors. Consider the perturbed system.

$$(A + \delta A)(x + \delta x) = b + \delta b \qquad (13)$$

A fundamental perturbation bound from numerical linear algebra yields

$$\frac{\| \delta x \|}{\| x \|} \leq \kappa(A) \frac{\| \delta A \|}{\| A \|} + \frac{\| \delta b \|}{\| b \|} + O(\epsilon^2) \qquad (14)$$

where ϵ denotes small relative perturbations. This bound clearly demonstrates that when κ(A) is large as is typical for nearly singular inverse problems tiny relative errors in either the data or the system matrix may be magnified into unacceptably large errors in the computed solution. This amplification mechanism is the fundamental source of instability in ill-posed inverse problems and motivates the need for stability analysis and regularization techniques.

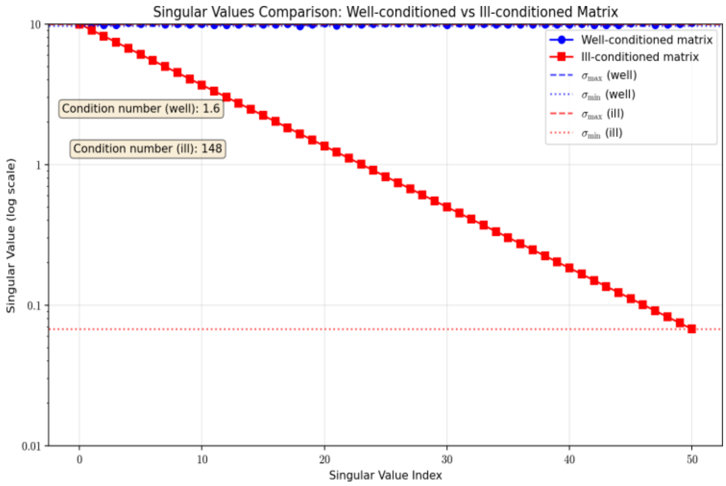

Fig. 1: Singular values composition: well-conditioned vs. ill-conditioned Source: created by the authors.

Figure 1 illustrating the distribution of singular values for a well-conditioned matrix and an ill conditioned matrix. The y-axis displays the singular value on a logarithmic scale, while the x-axis represents the singular value index. The singular values of the well-conditioned matrix (blue line) are closely grouped, leading to a condition number of 1.6. Conversely, the singular values of the ill-conditioned matrix (red line) exhibit a rapid decay, indicating a large spread and a significantly higher condition number of 148. The maximum (σ_{max}) and minimum (σ_{min}) singular values for each matrix are also shown.

2.5 SVD, Picard Condition, and Nearly Singular Matrices

The singular value decomposition (SVD)

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma u_i v_i^T \qquad (15)$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthonormal matrices containing the left and right singular vectors, respectively, and $\Sigma = diag(\sigma_1, \ldots, \sigma_r)$ contains the singular values ordered as
$\sigma_1 \geq \sigma_2 \geq \ldots, \geq \sigma_r > 0$, where $r = rank(A)$.
The minimum-norm least-squares solution can be written as a singular-value expansion involving terms of the form $(u_i^T y/\sigma_i)v_i$. When some σ_i are very small, the factors $1/\sigma_i$ amplify the components of y along the corresponding singular directions, making the solution extremely sensitive to noise.
The Picard condition [3] states that for a stable solution to exist without regularization, the coefficients $u_i^T y$ should decay at least as fast as σ_i. In nearly singular problems this condition is often violated, and noise dominates the higher-index components, which motivates the use of regularization [19]. In the numerical example later, we visualize the singular values and illustrate how they relate to observed instability.

3 Regularization as a Stabilization Technique

3.1 Tikhonov Regularization

Regularization stabilizes ill-posed problems by augmenting the data-fidelity term with a penalty that encodes prior information or desired properties of the solution. This is a widely used approach is Tikhonov regularization, which seeking a solution x_λ that minimizes a penalized least-square functional:

$$x_\lambda = \arg\min_x \{\| Ax - y \|_2{}^2 + \lambda^2 \| Lx \|_2{}^2\}, \qquad (16)$$

where, $\lambda > 0$ is the regularization parameter, and L is a regularization operator (e.g., identity matrix I for penalizing solution norm, or a discrete approximation of a derivative for penalizing solution smooth ness). The solution to this problem is given by the stabilized normal equations:

$$(A^T A + \lambda^2 L^T L)x_\lambda = A^T y$$

In SVD form with $L = I$, Tikhonov regularization introduces filter factors:

$$\phi(\lambda, \sigma_i) = \frac{\sigma_i{}^2}{\sigma_i{}^2 + \lambda^2} \qquad (17)$$

which damp the contributions corresponding to small singular values and thereby control noise amplification[20].

3.2 Other Regularization Techniques:

Beyond classical Tikhonov regularization, several alternative approaches have been developed to address the limitations of quadratic penalties in inverse problems [21]. Notable examples include truncated singular value decomposition (TSVD), sparsity-promoting regularization based on type penalties, and total variation (TV) regularization. These methods are capable of exploiting additional structural properties of the unknown solution, such as sparsity or piecewise smoothness, and have demonstrated strong performance in advanced imaging applications, including non-blind image deblurring and noise-adaptive reconstruction frameworks. In particular, variational and nonlinear regularization techniques have significantly extended the classical theory of inverse problems, providing improved stability and edge preservation compared to standard Tikhonov methods [22].

While the present study focuses on Tikhonov regularization to ensure analytical transparency and computational reproducibility, the underlying stability analysis and numerical framework can, in principle, be extended to more sophisticated regularization schemes, including sparsity- and total variation-based formulations.

3.3 Choosing the Regularization Parameter

A critical aspect of regularization is the selection of the regularization parameter λ. Various methods exist, including:

- **L-curve method:** Plots $log \| Ax_\lambda - y \|_2$ against $log \| Lx_\lambda \|_2$, and choosing λ near the corner of the L-curve (maximum curvature) often indicates a good compromise [17].
- **Generalized Cross-Validation (GCV):** Minimizes a score function that estimates the predictive error of the regularized solution [23].
- **Discrepancy Principle:** If the noise level $\| \varepsilon \|_2$ is known, λ is chosen such that the residual norm $\| Ax_\lambda - y \|_2$ matches the noise level [13].
- **Bayesian methods:** Interpreting λ as hyperparameter and use evidence maximization or hierarchical models to estimate it [24].

3.4 Characterizing Nearly Singular Matrices

A square matrix A is singular if its determinant is zero, meaning it does not have an inverse. For rectangular matrices, or numerically, a matrix is nearly singular if it behaves as if it were singular, specifically if it has singular values very close to zero. The singular value decomposition (SVD) of A is a powerful tool for understanding its properties:

$$A = U\Sigma V^T \quad (18)$$

where $U \in \mathbb{R}^{m\times n}$and $V \in \mathbb{R}^{n\times n}$are orthogonal matrices, and $\Sigma \in \mathbb{R}^{m\times n}$is a diagonal matrix con taining the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > 0$ (where $p = min(m,n)$). If $m \neq n$, some singular values are zero. A nearly singular matrix has one or more singular values σ_i that are very small relative to the largest singular value σ_1[25],[20].

When solving $x = A^{-1}b$ (assuming A is square and invertible) or the least squares problem $\min_x \| Ax - y \|_2$ for non-square A),the presence of small singular values leads to severe noise amplification. If b is perturbed by a small error δb, the resulting solution $x\delta = A^{-1}(b + \delta b)$ will have an error $\delta x = A^{-1}\delta b$. The components of δx corresponding to small singular values can become disproportion ately large, swamping the true solution [1].

4 Efficient Algorithms for Large-Scale Regularized problem

4.1 Computational Challenges

For multidimensional problems, the number of unknowns is often very large, and forming $A^TA + \lambda^2L^TL$ explicitly can be prohibitively expensive in memory and time. Instead, one exploits the structure of the forward operator and uses matrix-free methods that only require matrix–vector products with A and A^TA. Even then, the cost of each operator application can be significant, as in CT or large-scale image deblurring, which motivates careful algorithmic design [1], [19].

4.2 Preconditioned Iterative Solvers

Iterative methods such as conjugate gradient (CG) [26], LSQR (Least Squares QR) [27], and GMRES play a central role in solving large, sparse linear systems and least-squares problems, especially when direct factorization is impractical. Their convergence behavior is strongly influenced by the spectral properties and condition number of the associated system matrix. In particular, for the Tikhonov normal equations $(A^TA + \lambda^2L^TL)x_\lambda = A^Tb$, the resulting coefficient matrix is symmetric positive definite under mild assumptions, so CG can be applied directly.

Preconditioning is crucial in this context, because nearly singular or severely ill-conditioned systems can cause iterative methods to converge very slowly or even stagnate. A preconditioner PP is constructed so that the preconditioned system (for example $P^{-1}(A^TA + \lambda^2L^TL)$ has a reduced effective condition number and a more favorable eigenvalue distribution than the original matrix. Typical choices include incomplete factorizations, simple diagonal scaling, and problem-specific preconditioners that exploit the structure of the forward operator. In our MATLAB demonstration, we adopt a practical, matrix-free preconditioning strategy that can be applied using only matrix–vector products, making it suitable for large-scale multidimensional inverse problems [27].

For symmetric positive definite systems, preconditioned CG is particularly efficient, while for general non-symmetric or indefinite problems, methods such as GMRES or LSQR are more appropriate, with LSQR being especially well suited to large least-squares formulations typical of inverse problems.

The preconditioner (P) for a matrix (A) is a matrix such that (P^{-1}A) has a smaller condition number than (A). Often, ($T = P^{-1}$) is called the preconditioner, since (P) itself may not be explicitly available. Modern methods apply ($T = P^{-1}$) in a matrix-free way, meaning neither (P) nor (T) (and sometimes not even (A)) are stored as matrices; only their action on vectors is computed.

Preconditioners are used in iterative solvers for equation (1) to improve convergence by reducing the matrix's condition number. Preconditioned solvers are especially effective for large or sparse systems and can work when (A) is not stored explicitly. Preconditioning can be applied in three ways:

1. Right preconditioning: Solve $(AP^{-1})y = b)$ for (y), then $(Px = y)$.

5 Left preconditioning: Solve

2. Two-sided preconditioning: Solve $(QAP^{-1}(Px) = Qb)$ to preserve symmetry, often with diagonal scaling.

The goal is to reduce the condition number of $(P^{-1}A)$ or (AP^{-1}), improving convergence and stability. The preconditioner (P) should balance effectiveness (lower condition number) with low cost of applying (P^{-1}). For example:

- $(P = I)$ is cheap but does nothing.

- $(P = A)$ is optimal but as hard as solving the original system.

Preconditioned iterative methods are equivalent to standard iterative methods applied to the preconditioned system. For example, Richardson iteration becomes:

$$[x_{n+1} = x_n \gamma_n P^{-1}(Ax_n - b), n \geq 0.]$$

Common preconditioned iterative methods include the preconditioned conjugate gradient, biconjugate gradient, and generalized minimal residual (GMRES) methods.

6 MATLAB Demonstration: 2D Image Deblurring

6.1 Problem Setup

To illustrate the interplay between stability analysis, regularization, and efficient algorithms, we consider a classical 2D image deblurring problem. The true image x_{true} is blurred by a convolution operator A and contaminated with additive Gaussian noise e, yielding observed data

$$b = Ax_{true} + e \qquad (19)$$

In matrix form, A is large, sparse, and nearly singular because it strongly attenuates high-frequency components of the image. Attempting to invert A or to solve the least-squares problem without regularization leads to pronounced noise amplification and visually meaningless reconstructions.

6.2 Numerical Workflow

The numerical experiment proceeds as follows:

1. Generate a test image and define a Gaussian blurring operator implemented in matrix-free form.
2. Simulate blurred and noisy data bb.
3. Compute an unregularized solution to demonstrate instability and the effect of ill-conditioning.
4. For a range of λ values, solve the Tikhonov-regularized normal equations using preconditioned CG, recording residual norms and solution semi-norms.
5. Construct the L-curve and identify a parameter λ near to the corner.
6. Visualize the original image, blurred/noisy data, unregularized reconstruction, and regularized reconstructions for different λ.

6.3 Numerical Results and Stability Analysis

The behavior observed in the images and error measures directly reflects the underlying singular value distribution and condition number of the blurring operator. Small singular values correspond to high-frequency features that are strongly damped in the forward model but strongly amplified in naive inversion. Tikhonov regularization modifies the effective singular value spectrum via filter factors, while preconditioning improves the convergence properties of the iterative solver without changing the regularized solution itself.

Fig. 2: Original Image, Blurred Image, and Blurred with Noise Data, Source: [28].

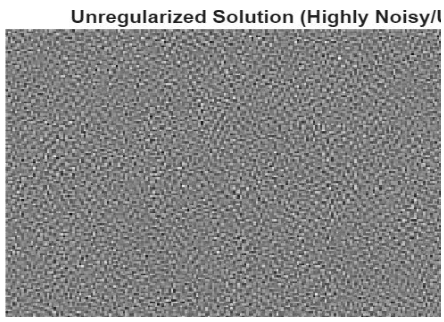

Fig. 3: Unregularized Solution (Highly Noisy and Unstable), Source: [29].

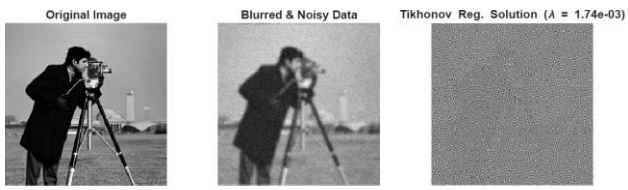

Fig. 4: Image Reconstruction Comparison, Source: [30].

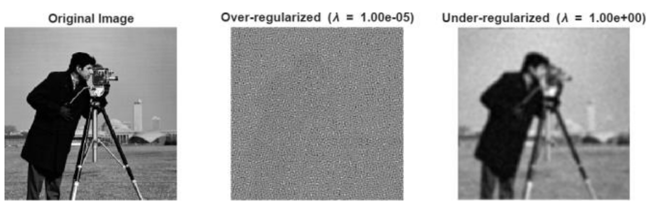

Fig. 5: Regularization Comparison, Source: [30].

Let's explain the overall Figure 2, Figure 3, Figure 4 and Figure 5 the output linking it back to the concepts of stability analysis and efficient algorithms for nearly singular multidimensional linear inverse problems.

In Figure 2, three display images are the **Original Image (x_{true}):** this is the ground truth we want to recover. It's sharp and clear. The second image, **Blurred Image (Ax_{true}):** This image shows the effect of the blurring operator A. Details are softened, and edges are less defined. This is what our forward model produces without noise. The third image, **Blurred and Noisy Data (b):** This is our actual input data. It's not only blurred but also contaminated with random noise. This b is what we feed into our inverse solver.

Figure 3, showing the unregularized solution (highly noisy/unstable), vividly illustrates the concept of ill conditioning and the instability of nearly singular inverse problems. When we try to solve for x directly or via a basic least-squares method (like lsqr without regularization terms in this context) on the noisy data b, the result is a chaotic, noise-dominated image.

Figure 4 illustrates a typical image restoration problem. The left image is the original, high-quality reference. The center image shows the degraded observation, blurred and corrupted by noise, mimicking an imperfect imaging system. The right image presents the result obtained using classical Tikhonov regularization. Despite using an optimized regularization parameter ($\lambda = 1.74 \times 10^{-3}$), the method fails to recover the original image. The reconstruction is dominated by noise and high-frequency artifacts, with blurred edges and visible ringing. This example highlights the limitations of standard Tikhonov regularization for complex image restoration tasks and motivates the use of more advanced regularization techniques to achieve meaningful results.

Figure 5 illustrates the effect of the regularization parameter λ on image reconstruction quality. Image regularization is a common technique used to solve ill-posed inverse problems, such as denoising or deblurring, by balancing fidelity to the observed data with a prior constraint on the solution's properties (e.g., smoothness). The original image, the leftmost panel, displays the ground-truth, high-quality image that serves as a reference. Over-regularized($\lambda = 1.00 \times 10^{-5}$), the central panel shows the result of using a very small regularization parameter. In this regime, the optimization process prioritizes fitting the data term almost exclusively. This causes the reconstruction to conform too closely to the noisy or degraded measurements, a phenomenon known as overfitting. The resulting image is dominated by high-frequency noise, and the underlying structure of the original scene is completely obscured. Under regularized($\lambda = 1.00 \times 100$), the right-most panel demonstrates the effect of a large regularization parameter. Here, the regularization term, which penalizes complexity, dominates the optimization. This forces the solution to be overly simple, failing to capture the details present in the original data. This is a case of underfitting, where the resulting image appears blurred and lacks fine textures and sharp edges.

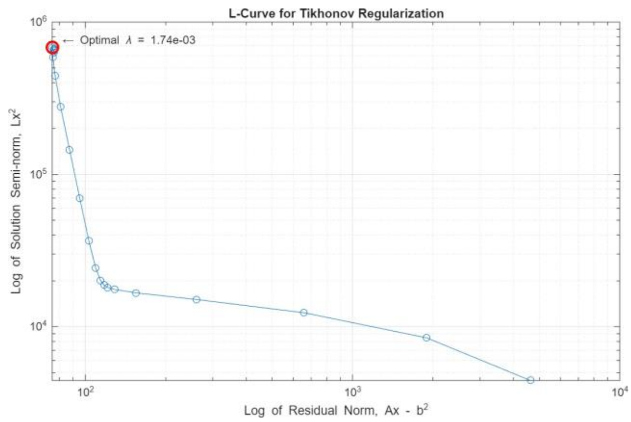

Fig. 6: L-Curve for Tikhonov Regularization
Source: created by the authors.

Figure 6 shows the L-curve used to select the regularization parameter λ in Tikhonov regularization. The plot displays the semi-norm solution against the residual norm for various λ values. Low λ values result in weak regularization and noise amplification, whereas high λ values lead to excessive smoothing and a poor fit to the data. The corner of the L-curve represents a balance between these two effects and is typically used to determine the optimal parameter value. In this case, the optimal value is $\lambda = 1.74 \times 10^{-3}$, that which provides a good balance between smoothness and data fidelity.

7 Result and Discussion

The results show that nearly singular multidimensional inverse problems can be solved effectively when stability and computational cost are balanced carefully. Stability analysis via the condition number and singular values explains why the blurring matrix A is problematic: it strongly damps high-frequency image content, while the noise in b spans all frequencies, so naive inversion amplifies small singular-value directions and produces unstable, meaningless solutions.

Tikhonov regularization stabilizes the problem by adding a penalty term that favors solutions with desired properties (small norm or smoothness), turning an ill-conditioned system into a well-conditioned one where small perturbations in bb cause only small changes in the regularized solution. For large 2D images, explicitly forming and inverting $(A^T A + \lambda^2 L^T L)$ is infeasible, so we solve the Tikhonov normal equations with conjugate gradient (CG), which only needs matrix–vector products and can exploit efficient convolution implementations. A simple diagonal preconditioner further improves convergence by reducing the effective condition number, significantly cutting the number of CG iterations.

Choosing $\boldsymbol{\lambda}$ correctly is crucial. The L-curve, plotting $\| Ax_\lambda - b \|^2$ versus $\| Lx_\lambda \|^2$, highlights three regimes:

- Under-regularization (small $\boldsymbol{\lambda}$): the solution overfits the noisy data, remains noisy and oscillatory, and has a large semi-norm.
- Over-regularization (large $\boldsymbol{\lambda}$): the solution is overly smooth, loses fine details, and shows a large data misfit.

  8 Optimal (corner of the Lcurve): the reconstruction preserves essential image features while suppressing most of the noise.

Visually, the Tikhonov solution at the L-curve corner is clearly better than the unregularized reconstruction: noise is strongly reduced and important structures are retained. Quantitatively, this is confirmed by higher PSNR values for the optimally regularized image compared to both the noisy data and the unregularized solution, indicating a reconstruction much closer to the true image. Overall, the demonstration confirms that an appropriately chosen $\boldsymbol{\lambda}$ is critical to balance noise suppression and detail preservation, avoiding both overfitting and oversmoothing.

9 Conclusion

Nearly singular multidimensional linear inverse problems represent a pervasive and challenging class of computational tasks across science and engineering. The inherent ill-conditioning of these problems necessitates a thorough understanding of stability issues, fundamentally quantified by the condition number and singular value analysis. This demonstration in MATLAB effectively showcases the critical interplay between stability analysis and efficient algorithms for nearly singular multidimensional linear inverse problems. Stability Analysis reveals the inherent ill conditioning of the problem, where small data perturbations are catastrophically amplified into noisy, unstable solutions. Tikhonov Regularization provides the mathematical framework to stabilize these problems by introducing prior knowledge or desired solution properties. Efficient Algorithms, specifically Preconditioned Conjugate Gradient (PCG) with function handles for large implicit operators, are indispensable for solving these high-dimensional, regularized systems within practical computational limits. The L-curve criterion guides the selection of an appropriate regularization parameter, balancing fidelity to data with stability of the solution. The computational demands of multidimensional scenarios man date the use of efficient numerical algorithms, specifically preconditioned iterative solvers. The synergy between robust regularization and carefully designed preconditions for iterative methods is not merely an optimization; it is a fundamental requirement for obtaining accurate, stable, and timely solutions to these critically important inverse problems. The result is a robust, stable, and computationally feasible reconstruction of the desired information (the original image) from incomplete and noisy observations, which is fundamental to countless applications in science and engineering. For future work, we continue developing more robust and computationally inexpensive methods for choosing the regularization parameter λ, especially for problems where multiple parameters or spatially varying regularization are desired. Continued research in these intertwined areas promises to unlock even more complex and impactful applications in the future.

References:

[1] Per Christian Hansen, *Discrete Inverse Problems: Insights and Algorithms*, SIAM, 2010.https://doi.org/10.1137/1.9780898718836.

[2] R. C. Aster, B. Borchers and C. H. Thurber, *Parameter Estimation and Inverse Problems*, Elsevier, 2019.https://doi.org/10.1016/C2015-0-02458-3.

[3] D. Calvetti, M. Dunlop, E. Somersalo and A. Stuart, Iterative Updating of Model Error

for Bayesian Inversion, Inverse Problems, Vol. 34, No. 2, 2018, Article 025008. https://doi.org/10.1088/1361-6420/aaa34d.

[4] K. Stamnes, W. Li, S. Stamnes, Y. Hu, Y. Zhou, N. Chen and J. Stamnes, A Novel Approach to Solve Forward/Inverse Problems in Remote Sensing Applications, *Frontiers in Remote Sensing*, Vol. 3, 2022, Article 1025447. https://doi.org/10.3389/frsen.2022.1025447.

[5] R. A. Willoughby, Solutions of Ill-Posed Problems (A. N. Tikhonov and V. Y. Arsenin), SIAM Review, Vol. 21, No. 2, 1979, pp. 266–267. https://doi.org/10.1137/1021044.

[6] J. Daoudi and C. Tajani, Genetic Algorithm-Based Optimization Approach for Solving a Class of Inverse Problems with Tikhonov Regularization, WSEAS Transactions on Mathematics, Vol. 22, 2023, pp. 92–102. https://doi.org/10.37394/23206.2023.22.

[7] C.-H. Huang and J.-Y. Yan, an Inverse Problem in Simultaneously Measuring Temperature-Dependent Thermal Conductivity and Heat Capacity, International Journal of Heat and Mass Transfer, 1995. https://doi.org/10.1016/0017-9310(95)00059-I.

[8] H. Meftahi and C. Nssibi, Stability Analysis of an Inverse Coefficients Problem in a System of Partial Differential Equations, Acta Applicandae Mathematicae, Vol. 203, 2026, Article 4. https://doi.org/10.1007/s10440-026-00793-7.

[9] J. C. Liu, Reconstruction Algorithms of an Inverse Coefficient Identification Problem for the Schrodinger Equation, Mathematical Modelling and Analysis, Vol. 22, No. 3, 2017, pp. 352–372. https://doi.org/10.3846/13926292.2017.1312580.

[10] S. Gazzola, P. C. Hansen and J. G. Nagy, IR Tools: A MATLAB Package of Iterative Regularization Methods and Large-Scale Test Problems, Numerical Algorithms, Vol. 81, 2019, pp. 773–811. https://doi.org/10.1007/s11075-018-0570-7.

[11] S. R. Gupta and P. K. Sahoo, Regularized Reconstruction and Performance Analysis in Inverse Imaging with PSNR and SSIM Metrics, *International Journal of Imaging Systems and Technology*, Vol. 33, No. 1, 2023, pp. 57–75. https://doi.org/10.1002/ima.22789.

[12] N. İ. Acar and A. Daşcioğlu, A Beneficial Numerical Approach to Solve Systems of Linear Integro-Differential Equations, *WSEAS Transactions on Systems and Control*, Vol. 19, 2024, pp. 309–315. https://doi.org/10.37394/23203.2024.19.32.

[13] V. Isakov, Inverse Problems for Partial Differential Equations, 3rd ed., Springer, 2017. https://doi.org/10.1007/978-1-4939-0790-8_3.

[14] V. A. Morozov, *Methods for Solving Incorrectly Posed Problems*, Springer-Verlag, 1984.https://doi.org/10.1007/978-1-4612-5280-1.

[15] A. N. Tikhonov, A. V. Goncharsky, V. V. Stepanov and A. G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, Kluwer Academic Publishers, 1995.https://doi.org/10.1007/978-94-015-8480-7.

[16] Zamani, M., Hladík, M. Error bounds and a condition number for the absolute value equations. Math. Program. 198, 85–113 (2023). https://doi.org/10.1007/s10107-021-01756-6.

[17] P. C. Hansen and D. P. O'Leary, The Use of the L-Curve in the Regularization of Discrete Ill-Posed Problems, *SIAM Journal on Scientific Computing*, Vol. 14, No. 6, 1993, pp. 1487–1503. https://doi.org/10.1137/0914086.

[18] J. Cheng and B. Hofmann, Regularization Methods for Ill-Posed Problems, in: O. Scherzer (ed.), Handbook of Mathematical Methods in Imaging, Springer, New York, NY, 2015. https://doi.org/10.1007/978-1-4939-0790-8_3.

[19] T. Raus and U. Hämarik, About the Balancing Principle for Choice of the Regularization Parameter, Journal of Physics: Conference Series, Vol. 135, 2008, 012087. https://doi.org/10.1088/1742-6596/135/1/012087.

[20] P. C. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion*, SIAM, 1998.https://doi.org/10.1137/1.9780898719697.

[21] A. Chambolle and T. Pock, A First-Order Primal–Dual Algorithm for Convex Problems with Applications to Imaging, *Journal of Mathematical Imaging and Vision*, Vol. 40, No. 1, 2011, pp. 120–145. https://doi.org/10.1007/s10851-010-0251-1.
[22] M. Benning and M. Burger, Modern Regularization Methods for Inverse Problems, *Acta Numerica*, Vol. 27, 2018, pp. 1-111. https://doi.org/10.1017/S0962492918000016.
[23] G. H. Golub, M. Heath and G. Wahba, Generalized Cross-Validation as a Method for Choosing a Good Ridge Parameter, *Technometrics*, Vol. 21, No. 2, 1979, pp. 215–223. https://doi.org/10.1080/00401706.1979.10489751.
[24] J. Kaipio and E. Somersalo, *Statistical and Computational Inverse Problems*, Springer, 2005. https://doi.org/10.1007/b138659.
[25] C. Xu and W. Xu, Fast Algorithms for Singular Value Decomposition and the Inverse of Nearly Low-Rank Matrices, *National Science Review*, Vol. 10, No. 6, 2023, Article nwad083. https://doi.org/10.1093/nsr/nwad083.
[26] C. C. Paige and M. A. Saunders, LSQR: An Algorithm for Sparse Linear Equations and Sparse Least Squares, *ACM Transactions on Mathematical Software*, Vol. 8, No. 1, 1982, pp. 43–71. https://doi.org/10.1145/355984.355989.
[27] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, 2003. https://doi.org/10.1137/1.9780898718003.
[28] P. C. Hansen, J. G. Nagy and D. P. O'Leary, *Deblurring Images: Matrices, Spectra, and Filtering*, SIAM, 2006. https://doi.org/10.1137/1.9780898718874.
[29] W. C. Karl, *Regularization in Image Restoration and Reconstruction*, in *Handbook of Image and Video Processing*, 2005, pp. 183–V, https://doi.org/10.1016/B978-012119792-6/50075-9.
[30] K. Zhang, W. Zuo, Y. Chen, D. Meng and L. Zhang, *Beyond a Gaussian Denoiser: Residual Learning of Deep CNN for Image Denoising*, *IEEE Transactions on Image Processing*, Vol. 26, No. 7, 2017, pp. 3142–3155. https://doi.org/10.1109/TIP.2017.2662206

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)
The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
The authors declare that no funding was received for conducting this research or for the preparation of this article.

Conflict of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.