

Recent Advances in the Nield-Kuznetsov Functions

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Abstract—In this article, we discuss a class of functions known as the Nield-Kuznetsov functions, introduced over the past decade. These functions arise in the solutions to inhomogeneous Airy's and Weber's equations. Derivations of these functions are provided, together with their methods of computations.

Keywords— Nield-Kuznetsov, Weber, Airy, Inhomogeneous

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1. Introduction

Many interesting equations of mathematical physics have their roots in the nineteenth century, and arose then due to their various applications in physics. Their solutions have been expressed in the forms of integral functions which gave rise to various special functions that we know today.

Two examples of the said equations are Airy's and Weber's differential equations. Both equations continue to receive considerable attention in the literature (*cf.* [1, 2, 3, 4] and the references therein) due to their various practical applications and theoretical implications. In addition to its direct applicability in the study of optics, fluid flow and electromagnetism, many other differential equations in mathematical physics can be reduced to Airy's or Weber's equations by an appropriate change of variables.

Their theoretical implications are found in the introduction of other functions in seeking their solutions, and infinite series analysis of their solutions and the functions that are expressible in terms of Airy's functions and Weber's functions (*cf.* [5, 6, 7] and the references therein). In addition, these functions contribute to the creation of knowledge in their extensions to large arguments and complex domains of validity, [8], and generalizations of these functions whenever possible and necessary, [9].

Computationally, Airy's, Weber's, and other related special functions continue to represent a challenge and give rise to the need for efficient computational algorithms, (*cf.* [10, 11, 12] and the references therein). In fact, a large number of scientists have devoted the better parts of their careers to provide us with the State-of-the-Art computational

methodologies of these special functions (*cf.* [13, 14, 15, 16, 17] and the references therein).

Although Airy's and Weber's equations started out in their homogeneous forms, it is now evident that many applications require these equations in their more general, inhomogeneous forms with forcing functions that can be either constant or variable. For instance, Airy's inhomogeneous equation proved to be a valuable tool in the analysis of flow over porous layers in the presence of a transition layer, wherein Nield and Kuznetsov [18] showed that Brinkman's equation, which governs the flow in the variable permeability transition layer, can be reduced to Airy's equation. Abu Zaytoon *et.al.* [19, 20] provided models of variable permeability porous layers that resulted in reducing the governing equations to the generalized Airy's equation and Weber's equation.

Applications of the inhomogeneous Airy's and Weber's equations underscore the fundamental importance of solving their initial and boundary value problems, and providing the necessary computational methodologies to evaluate their integral functions. To this end, Nield and Kuznetsov [18] found it convenient to define a new integral function in terms of Airy's functions and their integrals. This provided a methodology of handling the inhomogeneous Airy's equation with a constant forcing function.

Properties of this newly-introduced integral function have been studied by Hamdan and Kamel, [21] and [22], who also introduced an integral function to express the particular solution of Airy's inhomogeneous equation with a variable forcing function. Integral functions suitable for expressing particular solutions of the generalized Airy's equation and the generalized Weber's equation were introduced by Abu Zaytoon *et.al.* [19], [20], and Alzahrani *et.al.* [23], and

analyzed extensively by the same authors (*cf.* [24, 25, 6, 27, 28, 29, 30, 31]).

The current work stems from our belief that knowledge of the equations of mathematical physics, and their associated solutions in the forms of integral and special functions, is an important aspect of modern research. These functions represent our mathematical treasure, and their computations are an integral part of our methodologies to expand this knowledge.

While we feel this is just the tip of the iceberg when it comes to advancing knowledge from homogeneous to inhomogeneous equations, and that there are many other equations to consider, our goal here is to provide this most up to date and comprehensive knowledge of six functions that have been developed over the past decade. Three classes of the Niid-Kuznetsov functions are considered (the Standard, the Generalized, and the Parametric Niid-Kuznetsov functions), each of which is of two kinds: the first and second kinds. They cover the two cases of inhomogeneities: a constant forcing function (Niid-Kuznetsov functions of the first kind) and variable forcing functions (Niid-Kuznetsov functions of the second kind).

In this work, we provide details of their derivations, and their ascending series representations that are suitable for efficient computations. Some sample computations are provided for illustration.

2. The Standard Niid-kuznetsov Functions

2.1 Initial Value Problem

Consider the inhomogeneous Airy's ordinary differential equation (ODE):

$$\frac{d^2y}{dx^2} - xy = f(x) \tag{1}$$

subject to the initial conditions

$$y(0) = \alpha \tag{2}$$

$$\frac{dy}{dx}(0) = \beta \tag{3}$$

where α and β are known values.

Solutions to (1) depend on the forcing, $f(x)$, which gives rise to the following four cases.

2.2 The Case of $f(x) \equiv 0$.

In this case, general solution to (1) is given by

$$y = a_1A_i(x) + a_2B_i(x) \tag{4}$$

where a_1 and a_2 are arbitrary constants, and the functions $A_i(x)$ and $B_i(x)$ are two linearly independent functions known as Airy's homogeneous functions of the first and

second kind, respectively, and are defined by the following integrals, [4]:

$$A_i(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{t^3}{3}\right) dt \tag{5}$$

$$B_i(x) = \frac{1}{\pi} \int_0^\infty [\sin\left(xt + \frac{t^3}{3}\right) + \exp\left(xt - \frac{t^3}{3}\right)] dt \tag{6}$$

The non-zero Wronskian of $A_i(x)$ and $B_i(x)$ is given by, [4]:

$$W(A_i(x), B_i(x)) = A_i(x) \frac{dB_i(x)}{dx} - B_i(x) \frac{dA_i(x)}{dx} = \frac{1}{\pi} \tag{7}$$

Solution (4) satisfying the initial conditions (2) and (3) takes the form

$$y = \pi \left\{ \frac{\frac{1}{6}\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{(3)^{\frac{1}{6}}\Gamma(\frac{2}{3})} \right\} A_i(x) + \pi \left\{ \frac{\alpha}{(3)^{\frac{1}{3}}\Gamma(\frac{1}{3})} + \frac{\beta}{(3)^{\frac{2}{3}}\Gamma(\frac{2}{3})} \right\} B_i(x) \tag{8}$$

where $\Gamma(\cdot)$ is the Gamma function.

2.3 The Case of $f(x) = \mp \frac{1}{\pi}$.

In these cases, general solution is given by a linear combination of Airy's functions and Scorer functions, [4], [6], as follows.

For $f(x) = -\frac{1}{\pi}$, general solution of (1) is given by

$$y = b_1A_i(x) + b_2B_i(x) + G_i(x) \tag{9}$$

where b_1 and b_2 are arbitrary constants, and

$$G_i(x) = \frac{1}{\pi} \int_0^\infty \sin\left(xt + \frac{t^3}{3}\right) dt \tag{10}$$

Solution to the initial value problem (1), (2), (3), in this case takes the form

$$y = \pi \left\{ \frac{\frac{1}{6}\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{(3)^{\frac{1}{6}}\Gamma(\frac{2}{3})} \right\} A_i(x) + \pi \left\{ \frac{\alpha}{(3)^{\frac{1}{3}}\Gamma(\frac{1}{3})} + \frac{\beta}{(3)^{\frac{2}{3}}\Gamma(\frac{2}{3})} - \frac{1}{3\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \right\} B_i(x) + G_i(x) \tag{11}$$

For $f(x) = \frac{1}{\pi}$, general solution of (1) is given by

$$y = c_1A_i(x) + c_2B_i(x) + H_i(x) \tag{12}$$

where c_1 and c_2 are arbitrary constants, and

$$H_i(x) = \frac{1}{\pi} \int_0^\infty \exp\left(xt - \frac{t^3}{3}\right) dt \tag{13}$$

Solution to the initial value problem (1), (2), (3), in this case takes the form

$$y = \pi \left\{ \frac{(3)^{\frac{1}{6}}\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{(3)^{\frac{1}{6}}\Gamma(\frac{2}{3})} \right\} A_i(x) + \pi \left\{ \frac{\alpha}{(3)^{\frac{1}{3}}\Gamma(\frac{1}{3})} + \frac{\beta}{(3)^{\frac{2}{3}}\Gamma(\frac{2}{3})} - \frac{4}{(3)^{\frac{2}{3}}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \right\} B_i(x) + H_i(x) \quad (14)$$

In applying the initial conditions (2) and (3) to obtain values of the arbitrary constants, we relied on values of the integral functions $A_i(x), B_i(x), G_i(x), H_i(x)$ and their first derivatives at $x = 0$, reported in [4]. These values are listed Tables 1(a) and 1(b), below.

TABLE 1(a)

Values of Airy's and Scorer functions at $x = 0$
$A_i(0) = \sqrt{3}G_i(0) = \frac{\sqrt{3}}{(3)^{\frac{7}{6}}\Gamma(\frac{2}{3})}$
$B_i(0) = 3G_i(0) = \frac{3}{(3)^{\frac{7}{6}}\Gamma(\frac{2}{3})}$
$G_i(0) = \frac{A_i(0)}{\sqrt{3}} = \frac{1}{(3)^{\frac{7}{6}}\Gamma(\frac{2}{3})}$
$H_i(0) = \frac{2B_i(0)}{3} = \frac{2}{(3)^{\frac{7}{6}}\Gamma(\frac{2}{3})}$

TABLE 1(b)

Values of the first derivatives of Airy's and Scorer functions at $x = 0$
$\frac{dA_i(0)}{dx} = -\sqrt{3} \frac{dG_i(0)}{dx} = \frac{-\sqrt{3}}{(3)^{\frac{5}{6}}\Gamma(\frac{1}{3})}$
$\frac{dB_i(0)}{dx} = 3 \frac{dG_i(0)}{dx} = \frac{3}{(3)^{\frac{5}{6}}\Gamma(\frac{1}{3})}$
$\frac{dG_i(0)}{dx} = -\frac{1}{\sqrt{3}} \frac{dA_i(0)}{dx} = \frac{1}{(3)^{\frac{5}{6}}\Gamma(\frac{1}{3})}$
$\frac{dH_i(0)}{dx} = \frac{2}{3} \frac{dB_i(0)}{dx} = \frac{2}{(3)^{\frac{5}{6}}\Gamma(\frac{1}{3})}$

2.4 The Case of $f(x) = R = \text{constant}$

In Case 2, above, the Scorer functions furnish the particular solution to inhomogeneous Airy's ODE for the special values $f(x) = \mp \frac{1}{\pi}$. If $f(x) = R$ is any constant or variable function of x , the Scorer functions do not directly render the needed particular solution.

When $f(x) = R$, a particular solution to ODE (1) is obtained using the method of variation of parameters to obtain the following general solution:

$$y = d_1 A_i(x) + d_2 B_i(x) - \pi R N_i(x) \quad (15)$$

where d_1 and d_2 are arbitrary constants, and

$$N_i(x) = A_i(x) \int_0^x B_i(t) dt - B_i(x) \int_0^x A_i(t) dt \quad (16)$$

The integral function $N_i(x)$ is called the *Standard Niels-Kuznetsov Function of the First Kind*. Niels and Kuznetsov, [18], introduced (16) in their study of the effect of transition layer in the flow over porous layers. Hamdan and Kamel [21] established the following relationships of $N_i(x)$ to Airy's and Scorer's functions:

$$N_i(x) = G_i(x) - \frac{1}{3} B_i(x) \quad (17)$$

$$N_i(x) = \frac{2}{3} B_i(x) - H_i(x) \quad (18)$$

$$N_i(x) = \frac{2}{3} G_i(x) - \frac{1}{3} H_i(x) \quad (18)$$

Using (10), (13) and (19), we obtain the following integral representation of $N_i(x)$:

$$N_i(x) = \frac{2}{3\pi} \int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt - \frac{1}{3\pi} \int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt \quad (20)$$

First and second derivatives of $N_i(x)$ are given by

$$\frac{dN_i(x)}{dx} = \frac{dA_i(x)}{dx} \int_0^x B_i(t) dt - \frac{dB_i(x)}{dx} \int_0^x A_i(t) dt \quad (21)$$

$$\frac{d^2N_i(x)}{dx^2} = -W(A_i(x), B_i(x)) = \frac{1}{\pi} \quad (22)$$

Equations (16) and (21) together with Tables 1(a) and 1(b), provide the following values at $x = 0$:

$$N_i(0) = \frac{dN_i(0)}{dx} = 0 \quad (23)$$

which are needed to evaluate the arbitrary constants, d_1 and d_2 , and write the solution to the initial value problem, (1), (2), (3), with $f(x) = R$, as

$$y = \pi \left\{ \frac{(3)^{\frac{1}{6}}\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{(3)^{\frac{1}{6}}\Gamma(\frac{2}{3})} \right\} A_i(x) + \pi \left\{ \frac{\alpha}{(3)^{\frac{1}{3}}\Gamma(\frac{1}{3})} + \frac{\beta}{(3)^{\frac{2}{3}}\Gamma(\frac{2}{3})} \right\} B_i(x) - \pi R N_i(x) \quad (24)$$

Remark 1.

We remark at the outset that in solving the initial value problem (1), (2), (3), and expressing the solution in terms of $N_i(x)$, values of the arbitrary constants are independent of the forcing function. To illustrate this point further, when

$$f(x) = \mp \frac{1}{\pi}, \text{ solution to the initial value problem is simply}$$

given by equation (24) with $R = \mp \frac{1}{\pi}$, while coefficients of

Airy's functions (values of the arbitrary constants) in the solution do not change. By comparison, when the general solution is expressed in terms of Scorer functions, $G_i(x)$ and $H_i(x)$, coefficients of Airy's functions do in fact change.

2.5 The Case of $f(x)$ Being General Function of x

When the forcing function is a general function of x , particular solution of (1) can be obtained using variations of parameters as $\pi K_i(x) - \pi f(x)N_i(x)$, and the general solution to (1) thus takes the form

$$y = e_1 A_i(x) + e_2 B_i(x) + \pi K_i(x) - \pi f(x)N_i(x) \quad (25)$$

where e_1 and e_2 are arbitrary constants, and the function $K_i(x)$, referred to as the *Standard Niels-Kuznetsov Function of the Second Kind*, takes the form

$$K_i(x) = A_i(x) \int_0^x \left\{ \int_0^t B_i(\tau) d\tau \right\} f'(t) dt - B_i(x) \int_0^x \left\{ \int_0^t A_i(\tau) d\tau \right\} f'(t) dt \quad (26)$$

wherein f' denotes the derivative of f with respect to its argument.

Equation (26) can be written in the following form in order to establish the connection between $K_i(x)$ and $N_i(x)$

$$K_i(x) = f(x)N_i(x) - \left\{ A_i(x) \int_0^x f(t)B_i(t) dt - B_i(x) \int_0^x f(t)A_i(t) dt \right\} \quad (27)$$

The following first derivative of $K_i(x)$ is obtained from (27):

$$\frac{dK_i(x)}{dx} = f'(x)N_i(x) + f(x) \frac{dN_i(x)}{dx} - \left\{ \frac{dA_i(x)}{dx} \int_0^x f(t)B_i(t) dt - \frac{dB_i(x)}{dx} \int_0^x f(t)A_i(t) dt \right\} \quad (28)$$

Equations (23), (27) and (28) furnish the values for $K_i(0)$ and $\frac{dK_i(0)}{dx}$:

$$K_i(0) = \frac{dK_i(0)}{dx} = 0 \quad (29)$$

Using conditions (2) and (3), we verify that the values of arbitrary constants e_1 and e_2 are the same as d_1 and d_2 , respectively, and the solution to the initial value problem (1), (2) and (3), when $f(x)$ is a variable function of x , is given by

$$y = \pi \left\{ \frac{\frac{1}{3}\alpha}{\Gamma(\frac{1}{3})} - \frac{\beta}{(\frac{1}{3})^6 \Gamma(\frac{2}{3})} \right\} A_i(x) + \pi \left\{ \frac{\alpha}{(\frac{1}{3})^3 \Gamma(\frac{1}{3})} + \frac{\beta}{(\frac{2}{3})^3 \Gamma(\frac{2}{3})} \right\} B_i(x) + \pi K_i(x) - \pi f(x)N_i(x) \quad (30)$$

Remark 2.

The function $N_i(x)$, defined in equation (16), is referred to as the *Standard Niels-Kuznetsov Function of the First Kind*. Particular solution to the inhomogeneous Airy's ODE with a constant forcing function is expressed in terms of $N_i(x)$. This function was introduced by Niels and Kuznetsov, [18]. Its extensive properties and relationship to other functions, such as Bessel functions, were studied in details and established in the work of Hamdan and Kamel, [21].

The function $K_i(x)$, defined in equation (27), is referred to as the *Standard Niels-Kuznetsov Function of the Second Kind*. Particular solution to the inhomogeneous Airy's ODE with a variable forcing function is expressed in terms of $K_i(x)$. This function was introduced by Hamdan and Kamel, [21].

2.6 Computations of the Standard Niels-Kuznetsov Functions

The standard Niels-Kuznetsov functions of the first and second kinds can be evaluated using asymptotic or ascending series (cf. [4, 18, 19, 20, 22, 26, 27]). Ascending series expressions are shown below, [26, 27]:

$$N_i(x) = 2\sqrt{3}a_1 a_2 \left[\left\{ \sum_{k=0}^{\infty} \binom{1}{3}_k \frac{3^k x^{3k}}{(3k)!} \right\} \left\{ \sum_{k=0}^{\infty} \binom{2}{3}_k \frac{3^k x^{3k+2}}{(3k+2)!} \right\} - \left\{ \sum_{k=0}^{\infty} \binom{1}{3}_k \frac{3^k x^{3k+1}}{(3k+1)!} \right\} \left\{ \sum_{k=0}^{\infty} \binom{2}{3}_k \frac{3^k x^{3k+1}}{(3k+1)!} \right\} \right] \quad (31)$$

$$K_i(x) = 2\sqrt{3}a_1 a_2 \left[\left\{ \sum_{k=0}^{\infty} \binom{1}{3}_k \frac{3^k x^{3k}}{(3k)!} \right\} \left\{ \sum_{k=0}^{\infty} \binom{2}{3}_k \int_0^x F(t) \frac{3^k t^{3k}}{(3k)!} dt \right\} - \left\{ \sum_{k=0}^{\infty} \binom{1}{3}_k \int_0^x F(t) \frac{3^k t^{3k-1}}{(3k-1)!} dt \right\} \left\{ \sum_{k=0}^{\infty} \binom{2}{3}_k \frac{3^k x^{3k+1}}{(3k+1)!} \right\} \right] \quad (32)$$

where

$$a_1 = A_i(0) \approx 0.3550280538878172$$

$$a_2 = -\frac{d}{dx} A_i(0) \approx 0.2588194037928067$$

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b(b+1)(b+2) \dots (b+k-1); k > 0$$

is the Pochhammer symbol, [7], where $(b)_0 = 1$.

Extensive tabulations have been carried out by Alzahrani et.al. [28, 29, 30, 31]. In Tables 2(a) and 2(b), below, are some values of $K_i(x)$ and $N_i(x)$ at selected values of x . Results were obtained using ascending series truncated after 10 terms.

TABLE 2(a)

$F(x)$	$K_i(x = 0.1)$	$K_i(x = 0.5)$	$K_i(x = 1)$
x	-0.0015916290	-0.040037971	-0.167256091
x^2	-0.0001061068	-0.013318270	-0.109689567
$\sin x$	-0.0015903031	-0.039213457	-0.154114269
e^x	-0.0334796769	-0.208436781	-0.584378919

TABLE 2(b)

$N_i(x = 0.1)$	$N_i(x = 0.5)$	$N_i(x = 1)$
-0.00159162909	-0.04003797119	-0.1672560919

3. The Generalized Niels-kuznetsov Functions

3.1 Solution to the Homogeneous Generalized Airy's Equation

Consider the generalized inhomogeneous Airy's ODE of the form:

$$\frac{d^2y}{dx^2} - x^n y = f(x) \quad (33)$$

In their elegant analysis, Swanson and Headley, [9], provided solution to the homogeneous part of (33) as a linear combination of the linearly independent functions $A_n(x)$, and $B_n(x)$, referred to as the generalized Airy's functions of the first and second kinds, respectively, namely:

$$y = a_1 A_n(x) + a_2 B_n(x) \quad (34)$$

where a_1 and a_2 are arbitrary constants and

$$A_n(x) = \frac{2p}{\pi} \sin(p\pi) (x)^{1/2} K_p(\zeta) \quad (35)$$

$$B_n(x) = (px)^{\frac{1}{2}} (I_{-p}(\zeta) + I_p(\zeta)) \quad (36)$$

The terms I_p and K_p are the modified Bessel functions defined as:

$$I_p(\zeta) = i^{-p} J_p(i\zeta) = \sum_{m=1}^{\infty} \frac{1}{m! \Gamma(m+p+1)} \left(\frac{\zeta}{2}\right)^{2m+p} \quad (37)$$

$$K_p(\zeta) = \frac{\pi (I_{-p}(\zeta) - I_p(\zeta))}{2 \sin(p\pi)} \quad (38)$$

with $p = \frac{1}{n+2}$, $\zeta = 2px$, and $\Gamma(\cdot)$ is the gamma function.

The Wronskian of $A_n(x)$ and $B_n(x)$ is given by:

$$W(A_n(x), B_n(x)) = \frac{2}{\pi} p^{\frac{1}{2}} \sin(p\pi) \quad (39)$$

and the first derivatives of the generalized Airy's functions are given by, [9]:

$$\frac{dA_n(x)}{dx} = -\frac{2p}{\pi} \sin(p\pi) (x)^{\frac{n+1}{2}} K_{p-1}(\zeta) \quad (40)$$

$$\frac{dB_n(x)}{dx} = p^{\frac{1}{2}} (x)^{\frac{n+1}{2}} [I_{1-p}(\zeta) + I_{p-1}(\zeta)] \quad (41)$$

Swanson and Headley [9] extensively studied properties of the generalized Airy's functions, $A_n(x)$, and $B_n(x)$, and their connections to other functions, such as Bessel functions.

In their modelling of flow through variable permeability porous layers, Abu Zaytoon *et.al.*, [19], showed that the flow equations could be reduced to the inhomogeneous Airy's ODE (33). In order to obtain solutions to initial value problems or boundary value problems, they solved (33) using variations of parameters and obtained the following cases of solutions.

3.2 The Case of Constant Forcing Function

When $f(x) = R = \text{constant}$, Abu Zaytoon *et.al.*, [19], showed that the general solution to (33) takes the form

$$y_n = b_1 A_n(x) + b_2 B_n(x) - \frac{\pi R}{2\sqrt{p} \sin(p\pi)} N_n(x) \quad (42)$$

where b_1 and b_2 are arbitrary constants, and

$$N_n(x) = A_n(x) \int_0^x B_n(t) dt - B_n(x) \int_0^x A_n(t) dt \quad (43)$$

with first derivative given by

$$\frac{dN_n(x)}{dx} = \frac{dA_n(x)}{dx} \int_0^x B_n(t) dt - \frac{dB_n(x)}{dx} \int_0^x A_n(t) dt \quad (44)$$

Remark 3.

Integral function (43) is termed the *Generalized Nield-Kuznetsov Function of the First Kind*, as introduced by Abu Zaytoon *et.al.* [19]. Clearly, when $n=1$, this function reduces to the Standard Nield-Kuznetsov Function of the First Kind, equation (16).

3.3 The Case of Variable Forcing Function

When the forcing function, $f(x)$, is a variable function of x , solution to (33) takes the form, [29]:

$$y_n = c_1 A_n(x) + c_2 B_n(x) - \frac{\pi}{2\sqrt{p} \sin(p\pi)} K_n(x) \quad (45)$$

where c_1 and c_2 are arbitrary constants, and

$$K_n(x) = B_n(x) \int_0^x F(t) \frac{dA_n(t)}{dt} dt - A_n(x) \int_0^x F(t) \frac{dB_n(t)}{dt} dt \quad (46)$$

The function $K_n(x)$ is referred to as the Generalized Nield-Kuznetsov Function of the Second Kind. Its first derivative is given by, [29]:

$$\frac{dK_n(x)}{dx} = \frac{dB_n(x)}{dx} \int_0^x F(t) \frac{dA_n(t)}{dt} dt - \frac{dA_n(x)}{dx} \int_0^x F(t) \frac{dB_n(t)}{dt} dt - \frac{2\sqrt{p} \sin(p\pi)}{\pi} F(x) \quad (47)$$

wherein $F' \equiv f$.

A. Computations of the Standard Nield-Kuznetsov Functions

Following Swanson and Headley, [9], the generalized Airy's functions are evaluated as follows.

Let $p = \frac{1}{n+2}$ and

$$\alpha_n = \frac{(p)^{1-p}}{\Gamma(1-p)} \quad \text{and} \quad \beta_n = \frac{(p)^p}{\Gamma(p)} \quad (48)$$

$$g_{n1}(x) = 1 + \sum_{k=1}^{\infty} p^{2k} \prod_{j=1}^k \frac{x^{(n+2)k}}{j(j-p)} \quad (49)$$

$$g_{n2}(x) = 1 + \sum_{k=1}^{\infty} p^{2k} \prod_{j=1}^k \frac{x^{(n+2)k}}{j(j-p)} \quad (50)$$

then

$$A_n(x) = \alpha_n g_{n1}(x) - \beta_n g_{n2}(x) \quad (51)$$

$$B_n(x) = \frac{1}{\sqrt{p}} [\alpha_n g_{n1}(x) + \beta_n g_{n2}(x)] \quad (52)$$

The generalized Nield-Kuznetsov functions can be evaluated using (41)-(44) and the following expressions, [19, 29, 30, 31]

$K_{10}(1)$	-0.02391327336	-0.01593053627
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$$N_n(x) = \frac{2}{\sqrt{p}} \alpha_n \beta_n \left\{ g_{n1}(x) \int_0^x g_{n2}(t) dt - g_{n2}(x) \int_0^x g_{n1}(t) dt \right\} \quad (53)$$

$$K_n(x) = \frac{-2}{\sqrt{p}} \alpha_n \beta_n \left\{ g_{n1}(x) \int_0^x F(t) \frac{d}{dt} g_{n2}(t) dt - g_{n2}(x) \int_0^x F(t) \frac{d}{dt} g_{n1}(t) dt \right\} \quad (54)$$

Upon evaluating (51), (52), (43), (44), (46), and (47) at $x = 0$, we obtain the following values in Table 3..

TABLE 3

Values of Generalized Airy's and Nield-Kuznetsov Functions, and Derivatives at Zero
$A_n(0) = \frac{(p)^{1-p}}{\Gamma(1-p)}$
$B_n(0) = \frac{(p)^{1/2-p}}{\Gamma(1-p)}$
$\frac{dA_n(0)}{dx} = -\frac{(p)^p}{\Gamma(p)}$
$\frac{dB_n(0)}{dx} = -\frac{(p)^{p-1/2}}{\Gamma(p)}$
$N_n(0) = 0$
$\frac{dN_n(0)}{dx} = 0$
$K_n(0) = 0$
$\frac{dK_n(0)}{dx} = -\frac{2\sqrt{p} \sin(p\pi)}{\pi} F(0)$

Computations of the generalized Nield-Kuznetsov functions, using (53) and (54), have been carried out and tabulated by Alzahrani et.al. [30], for $n=1,2,\dots,10$. It is worth noting that (54) reduces to (53) when $F(x) = x$, and $x = 1$, as verified by values of $N_n(1)$ and $K_n(1)$ in Tables 4 and 5 below.

TABLE 4

Values of $N_n(1)$ for $n = 1, 2, 3, 4, 5, \text{ and } 10$
$N_1(1) = -0.1671679498$
$N_2(1) = -0.1163218192$
$N_3(1) = -0.08567790804$
$N_4(1) = -0.06614082173$
$N_5(1) = -0.05292832891$
$N_{10}(1) = -0.02391327336$

TABLE 5

Values of $K_n(1)$	$F(x) = x$	$F(x) = x^2$
$K_n(1)$	$F(x) = x$	$F(x) = x^2$
$K_1(1)$	-0.1671679498	-0.1095475388
$K_2(1)$	-0.1163218194	-0.07681056585
$K_3(1)$	-0.08567790816	-0.05678057772
$K_4(1)$	-0.06614082166	-0.04391997436
$K_5(1)$	-0.05292832894	-0.03518808431

4. The Parametric Nield-kuznetsov Functions

A. 4.1 Weber's Homogeneous Equation

Weber's inhomogeneous differential equation finds applications in the study of fluid flow through porous layers with variable permeability, [20]. Weber's homogeneous equation takes the form

$$\frac{d^2y}{dx^2} + (ax^2 + bx + c)y = 0 \quad (55)$$

where a, b, c are constants and the independent variable, x , is in general complex.

However, it has been reported in the literature in other forms, (cf. [1, 2, 3]) that include three distinctive forms whose mathematical and physical applications have been discussed in the work of Temme, [2]. Of interest, due to its validity in the real domain and its usefulness in modelling practical situations in the real plain, is Weber's equation of the form

$$\frac{d^2y}{dx^2} + \left(\frac{x^2}{4} - a\right)y = 0 \quad (56)$$

Equation (56) possesses the solutions $W(a, \mp x)$, where the variable x and the parameter a are real numbers. These solutions represent a linearly independent, numerically satisfactory pair of solutions, [2], for all $x \in \mathfrak{R}$, with the Wronskian given by

$$w(W(a, x), W(a, -x))=1. \quad (57)$$

Abu Zaytoon *et.al.* [20] encountered applications in the study of flow through porous layers, of the inhomogeneous Weber's equation, namely:

$$\frac{d^2y}{dx^2} + \left(\frac{x^2}{4} - a\right)y = f(x) \quad (58)$$

and showed that a particular solution is given by

$$y_p = W(a, -x) \int_0^x f(t)W(a, t)dt - W(a, x) \int_0^x f(t)W(a, -t)dt \quad (59)$$

The following two cases arise, depending on $f(x)$.

4.2 The Case of $f(x) = R = \text{constant}$

If $f(x) = R$, then

$$y_p = R\{W(a, -x) \int_0^x f(t)W(a, t)dt - W(a, x) \int_0^x f(t)W(a, -t)dt\} \quad (60)$$

In equation (60), the expression in brackets is termed the *Parametric Nield-Kuznetsov Function of the First Kind*, which depends on parameter a , and is denoted and given by

$$N_w(a, x) = W(a, x) \int_0^x W(a, -t) dt - W(a, -x) \int_0^x W(a, t) dt \quad (61)$$

with a first derivative given by

$$\frac{dN_w(a, x)}{dx} = \frac{dW(a, x)}{dx} \int_0^x W(a, -t) dt + \frac{dW(a, -x)}{dx} \int_0^x W(a, t) dt \quad (62)$$

General solution to (58) can thus be written as:

$$y = c_1 W(a, x) + c_2 W(a, -x) - RN_w(a, x) \quad (63)$$

where c_1 and c_2 are arbitrary constants.

4.3 The Case of $f(x)$ Being a Variable Function of x

If $f(x)$ is a variable function of x , then

$$y_p = W(a, -x) \int_0^x F'(t) W(a, t) dt - W(a, x) \int_0^x F'(t) W(a, -t) dt \quad (64)$$

where $F'(t) = f(t)$.

Using integration by parts, equation (64) can be written as:

$$y_p = -\{W(a, -x) \int_0^x F(t) \frac{dW(a, t)}{dt} dt + W(a, x) \int_0^x F(t) \frac{dW(a, -t)}{dt} dt\} \quad (65)$$

The expression on the right-hand-side of (53) depends on parameter a , and is reminiscent of the Nield-Kuznetsov function of the second kind. It is referred to as the *Parametric Nield-Kuznetsov Function of the Second Kind*, denoted by $K_w(a, x)$, and defined as:

$$K_w(a, x) = W(a, -x) \int_0^x F(t) \frac{dW(a, t)}{dt} dt + W(a, x) \int_0^x F(t) \frac{dW(a, -t)}{dt} dt \quad (66)$$

with first derivative given by

$$\frac{dK_w(a, x)}{dx} = \frac{dW(a, x)}{dx} \int_0^x F(t) \frac{dW(a, -t)}{dt} dt - \frac{dW(a, -x)}{dx} \int_0^x F(t) \frac{dW(a, t)}{dt} dt - F(x) \quad (67)$$

General solution to (58) can thus be written as:

$$y = d_1 W(a, x) + d_2 W(a, -x) - K_w(a, x) \quad (68)$$

where d_1 and d_2 are arbitrary constants.

In solving initial and boundary value problems involving the Weber ODE, one requires values of the parametric Nield-Kuznetsov functions at given values of the argument. Values at $x = 0$ of $W(a, x)$, $\frac{dW}{dx}(a, x)$ are given by, [14, 15, 17]:

$$W(a, 0) = \frac{1}{(2)^{\frac{3}{4}}} \left| \frac{\Gamma(\frac{1}{4} + \frac{1}{2}ia)}{\Gamma(\frac{3}{4} + \frac{1}{2}ia)} \right|^{\frac{1}{2}} \quad (69)$$

$$W(0, 0) = \frac{1}{(2)^{\frac{3}{4}}} \left| \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \right|^{\frac{1}{2}} \quad (70)$$

$$\frac{dW}{dx}(a, 0) = -\frac{1}{(2)^{\frac{1}{4}}} \left| \frac{\Gamma(\frac{3}{4} + \frac{1}{2}ia)}{\Gamma(\frac{1}{4} + \frac{1}{2}ia)} \right|^{\frac{1}{2}} \quad (71)$$

$$\frac{dW}{dx}(a, 0) = -\frac{1}{(2)^{\frac{1}{4}}} \left| \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right|^{\frac{1}{2}} \quad (72)$$

Values of the parametric Nield-Kuznetsov functions and derivatives at $x = 0$ are provided in Table 6, below.

TABLE 6

Parametric Nield-Kuznetsov of first kind and derivative at zero	Parametric Nield-Kuznetsov of second kind and derivative at zero
$N_w(a, 0) = 0$	$K_w(a, 0) = 0$
$\frac{dN_w}{dx}(a, 0) = 0$	$\frac{dK_w}{dx}(a, 0) = -F(0)$

4.4 Computations of the Parametric Nield-Kuznetsov Functions

The following expressions, developed in [13-17], are used in the computations of the parabolic cylindrical functions $W(a, x)$ and $W(a, -x)$:

$$W(a, x) = W(a, 0) \sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n}}{(2n)!} + \frac{dW}{dx}(a, 0) \sum_{n=0}^{\infty} \delta_n(a) \frac{x^{2n+1}}{(2n+1)!} \quad (73)$$

$$W(a, -x) = W(a, 0) \sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n}}{(2n)!} - \frac{dW}{dx}(a, 0) \sum_{n=0}^{\infty} \delta_n(a) \frac{x^{2n+1}}{(2n+1)!} \quad (74)$$

$$\frac{d}{dx} W(a, x) = W(a, 0) \sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n-1}}{(2n-1)!} + \frac{dW}{dx}(a, 0) \sum_{n=0}^{\infty} \delta_n(a) \frac{x^{2n}}{(2n)!} \quad (75)$$

$$\frac{d}{dx} W(a, -x) = -W(a, 0) \sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n-1}}{(2n-1)!} + \frac{dW}{dx}(a, 0) \sum_{n=0}^{\infty} \delta_n(a) \frac{x^{2n}}{(2n)!} \quad (76)$$

Using (73)-(76), Alzahrani *et al* [23, 28], developed the following series expressions for the parametric Nield-Kuznetsov functions:

$$N_w(a, x) = 2W(a, 0) \frac{dW}{dx}(a, 0) \left[\left\{ \sum_{n=0}^{\infty} \delta_n(a) \frac{x^{2n+1}}{(2n+1)!} \right\} \left\{ \sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n+1}}{(2n+1)!} \right\} - \left\{ \sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n}}{(2n)!} \right\} \left\{ \sum_{n=0}^{\infty} \delta_n(a) \frac{x^{2n+2}}{(2n+2)!} \right\} \right] \quad (77)$$

$$K_w(a, x) = 2W(a, 0) \frac{dW}{dx}(a, 0) \left[\sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n}}{(2n)!} \int_0^x \delta_n(a) F(t) \frac{t^{2n}}{(2n)!} dt \right] \quad (78)$$

where

$$\rho_{n+2} = a\rho_{n+1} - \frac{1}{2}(n+1)(2n+1)\rho_n \quad (79)$$

$$\delta_{n+2} = a\delta_{n+1} - \frac{1}{2}(n+1)(2n+3)\delta_n \quad (80)$$

$$\rho_0(a) = \delta_0(a) = 1 \quad (81)$$

$$\rho_1(a) = \delta_1(a) = a \quad (82)$$

Sample calculations of $N_w(a, x)$ and $K_w(a, x)$ using the (77)-(82) are shown in Tables 7, 8 and 9, below.

TABLE 7

x	Some Values of $N_w(a, x)$	Some Values of $N_w(a, x)$
	$N_w(a, x); a = 0$	$N_w(a, x); a = 1$
0	0	0
0.1	-0.004999995827	-
		0.005004163888
0.2	-0.01999973332	-0.02006648827
0.3	-0.04499696254	-0.04533545954
0.4	-0.07998293445	-0.08105513505
0.5	-0.1249349069	-0.1275598486
0.6	-0.1798056698	-0.1852664744
0.7	-0.2445101224	-0.2546639319
0.8	-0.3189089751	-0.3362995397
0.9	-0.4027896942	-0.4307617641
1	-0.4958448910	-0.5386588510

TABLE 8

x	$K_w(a, x); a = 0$	$K_w(a, x); a = 1$
	$F(x) = x^2$	$F(x) = x^2$
0	0	0
0.1	-0.000333317456	-
		0.0003360053955
0.2	-0.002666463495	-0.002752690054
0.3	-0.008996529010	-0.009659773517
0.4	-0.02130733755	-0.02415201443
0.5	-0.04154278194	-0.05041843662
0.6	-0.07155657245	-0.09422931145
0.7	-0.1130311074	-0.1635219131
0.8	-0.1673594862	-0.2690996788
0.9	-0.2354868836	-0.4254486950
1	-0.3177111593	-0.6516586206

TABLE 9

y	$K_w(a, x); a = 0$	$K_w(a, x); a = 1$
	$F(x) = \sin x$	$F(x) = \sin x$
0	0	0
0.1	-0.004995813904	-0.005033369365
0.2	-0.01993209438	-0.02053562388
0.3	-0.04464846700	-0.04772579586
0.4	-0.07885511745	-0.08867593176
0.5	-0.1221006021	-0.1463657018
0.6	-0.1737300984	-0.2247467511
0.7	-0.2328376690	-0.3288039768
0.8	-0.2982174230	-0.4645964625
0.9	-0.3683198455	-0.6392561548
1	-0.4412208948	-0.8609179616

Further computations and solutions to initial and boundary value problems involving the six Nield-Kuznetsov functions have been carried out by Alzahrani *et al* [23, 28].

5. Conclusions

In this work, we provided a complete overview of the recent developments and advances over the past decade of the three classes of the Nield-Kuznetsov functions.

The Standard Nield-Kuznetsov functions of the first and second kinds arise due to the solution of Airy's inhomogeneous ODE with constant and variable forcing functions, respectively. They are defined by equations (16) and (27), respectively, and their ascending series expressions are given by equations (31) and (32), respectively.

The Generalized Nield-Kuznetsov functions of the first and second kinds arise due to the solution of generalized Airy's inhomogeneous ODE with constant and variable forcing functions, respectively. They are defined by equations (53) and (54), respectively, and their series expressions are given by equations (31) and (32), respectively.

The Parametric Nield-Kuznetsov functions of the first and second kinds arise due to the solution of Weber's inhomogeneous ODE with constant and variable forcing functions, respectively. They are defined by equations (61) and (66), respectively, and their series expressions are given by equations (77) and (78), respectively.

With this knowledge, one is now able to solve the inhomogeneous Airy's ODE, inhomogeneous generalized Airy's ODE and inhomogeneous Weber's ODE that involve initial value problems and two-point boundary value problems.

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