Time and band limiting: from the early days to the present

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Abstract: There are many situations in communications theory, medical imaging, geophysics, signal processing, and mathematics where one has an optimization problem whose solution is only rendered practical by some kind of mathematical miracle. A good and canonical example of this is the computation of the singular value decomposition for either a huge matrix or an integral operator. In particular these problems are typically extremely ill-posed. The work of D. Slepian, H. Landau and H. Pollak at Bell Labs 1960-1965 gives a remedy to this situation. Inspired in part by questions posed by Claude Shannon they found and exploited a miracle that allows for the effective computation of the so called "prolate spheroidal wave functions" which are defined as the eigenfunctions of an integral operator but turn out to be computable since they are also the eigenfunctions of a second order differential operator. The numerical computation of these functions has in this fashion become a stable problem, while the initial one was a very ill-posed one.

I will try to give an account of these developments and indicate at least one open problem inspired by this remarkable work at Bell Labs.

We will see that the original work started around 1960 has been extended in a few directions, and that the mathematical miracle underlying this work has influenced many other areas of mathematics ranging from the study of the Riemann zeta function to very recent work that is inspired by the same effort to find numerically stable ways to compute quantities of interest.

Key-Words: Time and band limiting, Limited angle tomography, Korteweg-deVries equation, Commuting integral and differential operators, Meixner-Pollaczek polynomials, Harmonic analysis

Received: June 15, 2024. Revised: November 24, 2024. Accepted: March 17, 2025. Published: May 9, 2025.

1 Introduction

Many years ago I was invited to give a talk at Bell Labs, in the presence of the authors of the main papers on the topic of "Time and band limiting". I was very happy since it was a huge audience, including lots of young people. I started by saying: "of course at Bell Labs I do not need to explain my title". Henry Landau, who was running the seminar interrupted me and said "listen Alberto, with the exception of the people in the first row, all of us with white hair, nobody here knows what you are talking about, so you better start from scratch". With this in mind I am going to start at the beginning.

All the developments below can be traced to the paper, [1].

Consider the finite Fourier transform

$$(Ef)(z) = \int_{-\tau}^{\tau} e^{izx} f(x)dx, \quad z \in [-\kappa, \kappa],$$

and the problem of determining its singular value decomposition.

In practical terms this leads to looking for the eigenfunctions of the integral operator

$$(EE^*f)(z) = 2\int_{-\kappa}^{\kappa} \frac{\sin \tau(z-w)}{z-w} f(w)dw,$$

for $z \in [-\kappa, \kappa]$.

It is well known that this is a numerically extremely unstable problem. A piece of magic intervenes here: back in the $1960 - 1964$, [2], [3], noticed and fully exploited the fact that this integral operator commutes with the differential operator given by

$$R(z, \partial_z) = \partial_z(\kappa^2 - z^2)\partial_z - \tau^2 z^2.$$

This differential operator $R(z, \partial_z)$ happens to be the "radial part" of the Laplacian in prolate-spheroidal coordinates. Its eigenfunctions are thus eigenfunctions of the integral operator EE^*.

These eigenfunctions are known by the name the prolate-spheroidal functions.

The study, [4], considered the same problem in n-dimensions. It turns out that the integral operator

$$(\mathcal{E}f)(z) = \int_0^1 J_N(czw)\sqrt{czw} f(w)dw$$

with $J_N(x)$ the Bessel function of the first kind, happens to commute with

$$\partial_z(1-z^2)\partial_z - c^2z^2 + \frac{1/4-N^2}{z^2},$$

The study, [5], showed that the integral operator with the so-called Airy kernel

$$\frac{A(z)A'(w)-A'(z)A(w)}{z-w}$$

happens to commute with

$$\partial_z(\tau-z)\partial_z - z(\tau-z).$$

Higher order cases were constructed in [6], by use of the Darboux transformations. The finite Laplace transform was considered in a paper that I wrote with M. Bertero. The reader should also look at [7], as well as [5], [8], [9].

The examples discussed above, and many others, have something in common. They are built from a kernel

$$K_\psi(z,w) := \int_{\Gamma_1} \psi(x,z)\psi^\dagger(x,w)dx \tag{1}$$

where the functions ψ are eigenfunctions of a differential operator, and one looks for a commuting differential operator. For important numerical work see [10].

2 A closer look at the work of the Bell Labs group

The Bell Labs group wrote a series of papers under the title "Prolate spheroidal wave functions, Fourier analysis and uncertainty", [2], [3], [4], [11], [12]. Of these I, II and III, treat signals in continuous time. The paper IV in this series deals with the multidimensional case.

Paper V, [12], considers the case when the time series is given by discrete samples of a signal.

The Bell Labs group did not consider the purely finite case, treated in [13].

The operator

$$L_\nu = -\partial_x^2 + \frac{\nu^2-1/4}{x^2} \qquad x>0 \tag{2}$$

is a crucial tool in many areas of mathematics.

D. Slepian discovered that the eigenfunction of L_ν give rise to a kermel

$$K(z_1,z_2) \equiv \int_0^T f_\nu(x,z_1)f_\nu(x,z_2)dx \tag{3}$$

which allows for a commuting differential operator, namely

$$\mathbb{A}_\nu = -\partial_z(G^2-z^2)\partial_z + z^2T^2 + G^2\frac{\nu^2-1/4}{z^2}. \tag{4}$$

Most of the results below are obtained by trying to understand and extend the original work at Bell Labs, see [14].

For the most up-to-date work on this problem see, [15], [16], [17], [18] and the many references in these papers which we cannot include in this paper. My initial motivation for trying to extend the original work came the topic of limited angle tomography, [19] and [20].

3 The Korteweg-de Vries equation

This is the partial differential equation of water waves theory given by

$$u_t = u_{xxx} - 6uu_x$$

Many remarkable methods have been found to produce explicit solutions of this equation, a very unique situation among non-linear PDE's.

The reader can find a good guide to this material in [21], [22], as well as [23], [24] and [25].

One can look at this equation as something similar to the heat equation: it originated in one area of physics but it has invaded and enriched many areas of mathematics.

The KdV equation is part of a hierarchy of which the simplest one is

$$u_t = u_x.$$

Of course this is trivial to solve. Its solution is given by time translation of the initial data.

Keep in mind that translation is the structure behind Fourier analysis. One can look at KdV as giving rise to a sort of nonlinear Fourier transform.

We will see later the relevance of this to an extension of earlier work, as very nicely mentioned in [26].

4 The Darboux process

The "Darboux process" starts from the (Schroedinger) operator

$$L = -\partial_x^2 + V(x) \tag{5}$$

whose eigenfunctions are $\psi(x,z)$, and gives a new family of operators $\tilde{L}(t)$ with eigenfunctions to be given below.

Factorize L

$$L = \left(-\partial_x - \frac{\phi'(x)}{\phi(x)}\right)\left(\partial_x - \frac{\phi'(x)}{\phi(x)}\right)$$

with $\phi(x)$ an eigenfunction of L with zero eigenvalue. One has

$$\phi(x) = \phi^{(1)}(x) + t\phi^{(2)}(x)$$

with $\phi^{(1)}(x)$, $\phi^{(2)}(x)$ a basis of the two-dimensional space of eigenfunctions of L with eigenvalue 0. The eigenfunction $\phi(x)$ is denoted by

$$\phi(x, t).$$

The new operator $\tilde{L}$, denoted by $\tilde{L}(t)$, is

$$\tilde{L}(t) \equiv \left(\partial_x - \frac{\phi'(x,t)}{\phi(x,t)}\right)\left(-\partial_x - \frac{\phi'(x,t)}{\phi(x,t)}\right)$$

one gets

$$\tilde{L}(t) = L - 2\partial_x^2 \log \phi(x,t). \tag{6}$$

When $\psi(x, z)$ solves

$$L\psi(x,z) = z^2\psi(x,z)$$

one has

$$\tilde{L}(t)\left(\partial_x - \frac{\phi'(x,t)}{\phi(x,t)}\right)\psi(x,z) = z^2\left(\partial_x - \frac{\phi'(x,t)}{\phi(x,t)}\right)\psi(x,z)$$

showing that the eigenfunctions of $\tilde{L}(t)$ are

$$\left(\partial_x - \frac{\phi'(x,t)}{\phi(x,t)}\right)\psi(x,z). \tag{7}$$

This method has been widely used to extend the work of the Bell Labs group, and of course, in many other situations.

Notice that if we define a function $\theta(x)$ by

$$V(x) = -\frac{1}{4x^2} - 2\partial_x^2 \log \theta(x)$$

and write

$$\phi(x,t) = \frac{\tilde{\theta}(x,t)}{\theta(x)}$$

one gets

$$\tilde{L}(t) = -\partial_x^2 - \frac{1}{4x^2} - 2\partial_x^2 \log \tilde{\theta}(x,t).$$

The eigenfunctions of this new operator are

$$(1/z)(\partial_x - \partial_x \log \frac{\tilde{\theta}(x,t)}{\theta(x)})\psi(x,z).$$

5 The bispectral problem

This problem was motivated by trying to extend the results of the Bell Labs group, see, [27], and look at the comments in page 178 of that paper.

The sequence of functions

$$\begin{aligned}
\theta_0 =& 1,\\
\theta_1 =& x^{1/2},\\
\theta_2 =& x^2 + t_1,\\
\theta_3 =& \frac{3}{4}x^{9/2} + t_2 x^{1/2},
\end{aligned}$$

are given in [27], using the recursion

$$\theta'_{k+1}\theta_{k-1} - \theta_{k+1}\theta'_{k-1} = (2k-1)\theta_k^2.$$

The potentials $V_k(x)$ obtained from the functions above, see, [25] , go with the so called "master symmetries of Korteweg-de Vries".

The extension of the results in [2], [3], [4], [12], has taken us far away from signal processing. A much more complete picture of these developments can be seen in the references at the end of this paper.

6 The KdV hierarchy and its master symmetries

Both the so called KdV hierarchy as well as what is known as its master symmetries, play a very important role in the extensions mentioned above. The interested reader can consult the references at the end of the paper for these developments.

7 Commuting integral and differential operators

For background, before we consider an example, the reader may want to see the previous sections as well as [27].

Other useful references are, [2], [3], [4], [11], [12].

7.1 An example.

Start with

$$L = -\partial_x^2 - \frac{1}{4x^2} - 2\partial_x^2 \log \theta_1. \tag{8}$$

which is also given by

$$L = -\partial_x^2 - \frac{1}{4x^2} + \frac{1}{x^2} = -\partial_x^2 + \frac{3}{4x^2}.$$

One use of the Darboux method gives

$$L_2(t_1) = -\partial_x^2 - \frac{1}{4x^2} - 2\partial_x^2 \log \theta_2. \tag{9}$$

The new eigenfunction is

$$\widetilde{f_1}(x,z) = (1/z)(\partial_x - \partial_x \log \frac{\theta_2}{\theta_1}) f_1(x,z). \quad (10)$$

The new kernel is

$$K(z_1,z_2) \equiv \int_0^T \widetilde{f_1}(x,z_1)\widetilde{f_1}(x,z_2)dx, \quad (11)$$

which becomes

$$K(z_1,z_2) = \frac{2t_1 f_1(T,z_1) f_1(T,z_2)}{(t_1+T^2)Tz_1z_2} + \frac{z_1 f_1(T,z_1) f_2(T,z_2) - z_2 f_2(T,z_1) f_1(T,z_2)}{z_1^2 - z_2^2}.$$

It is easy to see that

$$\frac{f(x,z_1)\frac{\partial f(x,z_2)}{\partial_x} - \frac{\partial f(x,z_1)}{\partial_x} f(x,z_2)}{z_1^2 - z_2^2} \quad (12)$$

is a primitive for the product $f(x,z_1)f(x,z_2)$.

It is now easy to get a differential operator that commutes with the integral one given by the kernel above.

8 Almost final comments

The results in [15], [16], [17], [18], [29], deal with the large class of situations related to the KP hierarchy, we have only dealt here with the master symmetries of the KdV evolution equations. These are related to the Schroedinger (second order) differential operator. The paper, [28], considers solutions of the bispectral problem when this second order differential operator is replaced by a third order one. The possible existence of commuting pairs of integral and differential operators in this case is -to the best of knowledge- largely unexplored territory. For interesting examples and theoretical tools that could be applied in this case, see, [29].

The problem considered here is an extension in the case of higher dimensional Euclidean spaces of the work in [4]. In a paper with L. Longhi and M. Perlstadt one finds an excursion into the non-commutative situation that arises when Euclidean space is replaced by spheres. That study was motivated purely by mathematical reasons, and yet several years later, people working in geophysics found these results to have practical use, see, [30], [31].

9 A connection with the Riemann zeta function

Here the proper references are [32], as well as [33]. The work of A. Connes and collaborators has shown remarkable similarities between zeros of the Riemann zeta function in different regions and the spectra of appropriate extension of the prolate sheroidal wave operator, inside or outside of the usual interval.

10 An extremal problem

This sections aims at stressing the impact of the work of the Bell Labs group from the $1960's$ in areas very far removed from signal processing or communication engineering. The point is, once again, that the effective computation of certain quantities can be greatly facilitated when one is dealing with a differential operator naturally related to the original problem.

In the recent arXiv posting $2504.05205v1$ by A. Bondarenko, J. Ortega-Cerda, D. Radchenko and K. Seip the authors consider an extremal problem considered by L. Hörmander and Bernhardsson back in 1993.

The problem is to determine the function of smallest L^1 norm in the class of all functions of so called exponential type not larger than π and satisfying the condition $f(0) = 1$. The authors of the 1993 paper did not manage to determine the function explicitely.

The function in question is factorized in a certain form in terms of a function ϕ. This function is characterized in two different ways, either one of them serves to pin it down. The two characterizations involve a functional equation in one case and the fact that ϕ satisfies a certain second order differential equation. This should remind us of the differential equation that D. Slepian, H. Landau and H. Pollak found in the $1960s$.

But more is true: one finds here a specific commutation relation that holds for a certain two-parameter family of differential operators. A certain differential operator plays a crucil role since it commutes with any element of the Klein four-group acting on holomorphic functions in the punctured complex plane.

The authors are well aware of the similarity with the work of ”time-frequency” localization of the Bell Labs group. The authors refer to [33], for the evolution of this work and most appropriately say that while the commutativity in question is easy to verify once somebody points it out, it remains hard to find a conceptual reason as to why it should be there.

A conceptual explanation is also lacking in the original work at Bell Labs, in all of its extensions reviewed in this paper and in this new case. This remains as a nice challenge for the future.

11 An open problem

The Lie group $SL(2,R)$ of all real matrices of size two with nonzero determinant, plays a fundamental role in areas as (apparently) unrelated as Number Theory and Microwave Propagation.

The paper, [34], takes a step in the effort to look at the spherical functions connected with the group $SL(2,R)$ and its subgroup $SO(2)$. One of the results in that paper is a description of the polynomial eigenfunctions of the operator

$$\Delta_n = -y^2(\partial_x^2 + \partial_y^2) + iny\partial_x.$$

Here the index n is an integer.

One finds in [34], an explicit description of these polynomial eigenfunctions that features a sequence of orthogonal polynomials $q_j[k]$ defined by the recursion relation

$$q_j[k](x) = xq_{j-1}[k](x) - (j-1)(j+2k)q_{j-2}[k](x).$$

Here the parameter k takes the values $k = 0, 1, 2, \ldots$. These polynomials appear in connection with eigenfunctions of the "weight n-Laplacian" with eigenvalue $-k(k+1)$.

Notice that the corresponding orthogonality measure on the real line will be an even function.

In the case when $k = 0$ I looked for the orthogonality measure for a while, without much success. At some point I mentioned this problem to my colleague Olga Holtz, see, [35], who quickly managed to identify the first few moments of the measure and then with the help of the Encyclopedia of Integer Sequences noticed that their generating function is related to

$$1 + \tan(x).$$

Following this observation, each one of us realized that the orthogonality measure has a density given by

$$x/(2\sinh(x\pi/2)).$$

In all honesty, and since our methods were entirely different, I got this answer up to an incorrect multiplicative factor. My search relied on painful numerical work, and I had a programming typo.

At this point I had another piece of good luck. I run into my friend Jim Pitman and mentioned to him the appearance of this weight with some hyperbolic function in it. He immediately sent me a few references including a paper of his with Marc Yor, see, [36], as well as some ground breaking work of C.N. Morris, see [37]. After this conversation a long forgotten memory came back to me: in the great book by William Feller, the teacher of my teacher, see, [38], one finds a table of interesting densities and their characteristic functions. The last one in the table features some hyperbolic function.

It is interesting to look first at the case $k = -1/2$ which does not appear in [34].

The generating function of the moments is given by

$$1/\cos(x)$$

and the density of the orthogonality measure is

$$2/\cosh(x/2)$$

and most interestingly this is, after a change of variables, the celebrated arcsine weight connected with coin tossing given by

$$1/(y^{1/2}(1-y)^{1/2})$$

in the interval $[0,1]$, to which Feller devotes the entire chapter 3 of his first volumen, [38], on fluctuations in the coin tossing game.

Two comments unrelated to the observation above: in 2022 I wrote a paper where one finds a use of the Feynman-Kac technology to study the coin tossing game, and in a paper with L. Velazquez and J. Wilkening, there is a treatment of an appropriate quantum version of this problem, where the arcsine law is replaced by a pair of deltas at the end of the interval $[0,1]$.

The relation between these two densities mentioned above, one in $[0,1]$ and the other on the whole real line is given by

$$x = \log(y/(1-y))$$

or equivalently

$$y = 1/(1 + \exp(-x))$$

sometimes referred to as the sigmoid function.

In the paper by Morris, see [37], the relation between y and x is called "the natural observation". In the notation of [37], this case is denoted by $f_{1,0}$.

We move now to other values of k. It turns out that the measures corresponding to the cases of interest in conection with $SL(2,R)$, [34], i.e. the cases when $k = 0, 1, 2, ...$ are obtained by taking convolutions of this density with itself.

The corresponding generating function for the moments is given by

$$1/\cos(x)^{2(k+1)}$$

and the fact that we are taking powers of this function in the particular case $k = -1/2$ accounts for the the fact that the densities are obtained by convolutions of this basic density.

In the case $k = 0$ we get the density denoted by $f_{2,0} = f_{1,0} * f_{1,0}$ in [37], namely

$$x/(2\sinh(\pi x/2))$$

which we recognize as the density mentioned earlier.

It may be appropriate to notice the relations between the densities $f_{1,0}$ and $f_{2,0}$ and the Gamma function. One has

$$\pi/\cos(\pi/2x) = |\Gamma(1/2 + ix/2)|^2$$

and

$$\pi x/(2\sinh(\pi/2x)) = |\Gamma(1 + ix/2)|^2$$

A final word in connection with $SL(2, R)$: one can do all of this for real values of k greater or equal to $-1/2$ and obtain also a family of non-polynomial eigenfunctions, thus extending some of the results in [34] .

12 The Truska, Meixner, Pollaczek polynomials

Restricting to the case when the orthogonality measure is even, the Meixner polynomials satisfy the recursion relation

$$M_n(x) = xM_{n-1}(x)+(n-1)(k_2+(n-2)\kappa)M_{n-2}(x).$$

Here the quantities κ and k_2 appear explicitly, see (5.4), in the original paper of J. Meixner. The appropriate reference is [39].

When $\kappa = -1$ and $2k + 2 = -k_2$ we get the recursion given earlier for the polynomials $q_j[k](x)$.

In 1949-1950 F. Pollaczek published a few notes in C. R. Acad. Sci. Paris, where he introduced some polynomials that generalize the ultraspherical or Gegenbauer polynomials.

These polynomials satisfy the recursion

$$nP_n^\lambda(x) = 2xP_{n-1}^\lambda(x) - (n + 2\lambda - 2)P_{n-2}^\lambda(x).$$

They have a nice Hypergeometric expression

$$P_n^\lambda(x) = (2)\lambda_n/n!(i^n)_2F_1(-n, \lambda + ix, 2\lambda, 2).$$

Our insistence on even densities implies that the standard parameter ϕ that appears in more general expressions for the Pollaczek polynomials is chosen to be $\pi/2$. The appropriate reference is [40].

In fact F. Pollaczek rediscovered in 1950 the polynomials that J. Meixner had found in 1934.

Other people came to similar results too, see [41] and [42].

While preparing this note another piece of good luck came my way. A very nice paper, [43], reveals the fact that even Meixner had only rediscovered his polynomials. L. Truksa, had already discovered them a few years earlier. It is interesting to note that these papers have the word “hypergeometric” in their title. The appropriate references are [44], [45] and [46].

For more uses of these polynomials, see for instance [47] and [48].

These polynomials play a role in fields such as financial mathematics, giving rise to interesting martingales, [49] and [50].

With a bit of creativity one can consider the Meixner-Pollaczek polynomials as bispectral (with one operator of infinite order) and then ask the question if appropriate versions of ”time-and-band-limiting” in this case will exhibit the commutativity miracle that the Bell Labs group discovered and exploited more than sixty years ago.

It is of interest to notice that the polynomials of Meixner and Pollaczek have been used in subjects as (apparently) diverse as Financial Mathematics and the Harmonic analysis related to the Lie group $SU(1, 1)$. This group is isomorphic to the group $SL(2, R)$ mentioned above.

A final comment: I think that Slepian, Landau and Pollak, who started this game by looking at an important problem in communication theory, could be amused to see that over the years it has made contact with many other parts of mathematics and physics. For an extreme example, see [51].

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.