The Exact Solution of ARL using an Integral Equation on the DMEWMA Chart for the SAR(P)$_L$ Process for Mean Shift Detection

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Abstract: - The main objective of this investigation is to prove the explicit formula for the average run length (ARL) of the modified double exponentially weighted moving average (DMEWMA) control chart in the context of a seasonal regression process. The explicit formula is compared to the numerical integral equation (NIE) method, with the percentage of accuracy serving as the evaluation metric for both approaches. Furthermore, the performance of the DMEWMA control chart is assessed by calculating the standard deviation of the run length (SDRL) and the median run length (MRL). To demonstrate the design and application of the DMEWMA control chart, a comparison is conducted with the modified exponentially weighted moving average (MEWMA) and the exponentially weighted moving average (EWMA) control charts under conditions of mean process shift. Key metrics such as EARL, ESDRL, and EMRL are utilized to measure the performance of these control charts across several design parameter configurations. To evaluate the explicit formula's practical effectiveness, it is applied to real-world data, specifically the production index for biofuel gasohol 91, to detect process variations. This case study highlights the efficiency and reliability of the DMEWMA control chart in monitoring and identifying shifts in the process mean.

Key-Words: - Average Run Length, Seasonal Autoregressive, Double Modified Exponentially Weighted Moving Average, zero-state, change point detection, Integral equation.

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1 Introduction

Variation is inevitable in the industrial production process, regardless of how well the system is designed. Understanding and controlling this variation is essential for maintaining product quality and efficiency. Variation comes from many controllable and uncontrollable factors that affect the quality of the product. The causes of abnormal production processes can be machines, workers, management, and raw materials. Therefore, the concept of product quality control was created. Quality control means controlling products to be at standard levels, including activities related to preventing products from being defective or damaged during production. To minimize waste in the process, statistical quality control (SQC) tools have been implemented. These tools help identify and correct defects, ensuring better process efficiency and product quality. One statistical method for production process control is the control chart, which helps detect changes in the process. Identifying variations early, it enables quick corrective actions to prevent product defects. In addition, control charts are used to determine the capability of the production process. Analyzing control charts regularly allows manufacturers to improve the quality of their products constantly. Control charts help reduce the variation in the production process, resulting in consistent product quality, [1], [2].

Control charts are a fundamental tool in quality control and are extensively employed. The study [3] was the first to devise the Shewhart control chart in 1931, as revealed by the study of the evolution of control charts. The Cumulative Sum control chart (CUSUM), which was proposed by [4], was among the many control charts that were developed later. The Exponentially Weighted Moving Average control chart (EWMA) was subsequently proposed by [5]. This control chart is capable of accurately identifying minor process modifications. In the study [6], subsequently introduced the Modified Exponentially Weighted Moving Average control chart (MEWMA), which is also highly effective in identifying minor process modifications. Subsequently, the study [7], established the constants as general constants in order to generate a modified EWMA control chart. Recently, the study [8], proposed a double-modified exponentially weighted moving average (DMEWMA), which is a hybrid of the two MEWMA control charts.

The majority of observations are independent and normally distributed. Nevertheless, the false alarm rate may be influenced by the characteristics values of interest having a non-normal distribution or the observations being related to each other in the process variables in practice. For instance, the error is exponentially distributed, and the observations are related in a stationary time series. Economic and financial data contain these observations, [9]. The quality control process is examined in this research when the observations are correlated in a time series and the error is exponentially distributed, [10], [11], [12]. This pattern is frequently observed in a variety of data sets, including industrial data, equity data, stock returns, and health data. Nevertheless, the time series format can be either stationary or non-stationary.

The criteria for assessing the efficiency or sensitivity of detecting changes in the chart are primarily based on the Average Run Length (ARL). The ARL represents the average number of samples that remain within the control limits before the process indicates the first signal of being out of control. This is categorized into two states: when the process is under control (ARL_0) and when it is out of control (ARL_1). The control chart with the highest efficiency in identifying changes is identified by the lowest ARL_1 value.

In general, the average run length (ARL) is the metric used to compare the performance of control charts. There are numerous methods for calculating the average run length, including the Monte Carlo Simulation method, which produces precise results but necessitates a significant amount of processing time. Subsequently, numerous researchers proposed the Markov chain method for calculating the average run length, [13], [14]. However, the results derived from this method are approximate and lack convergence properties. An Approach to Numerical Integral Equation (NIE). The average route length is estimated using precise mathematical calculations, including the Gaussian Rule, Trapezoidal Rule, and Simpson's Rule. Nevertheless, there is no underlying theory of convergence, [15], [16]. In addition to the previously mentioned methods for determining the ARL value, recent developments in advanced mathematics have led to the creation of explicit formulas for calculating it. Researchers have extensively studied this approach and found that these explicit formulas are both highly accurate and computationally efficient, requiring significantly less processing time compared to traditional methods, [17], [18], [19], [20].

The authors are motivated to investigate the formula of average run length for the DMEWMA control chart for time series data and compare the efficiency of the formula of average run length with the numerical integral equation method, as indicated above. In addition, the efficacy of the DMEWMA, MEWMA, and EWMA control charts was evaluated at each level of change in the process mean.

2 Materials and Methods

2.1 Process

The sequences of Y_t in a $SAR(P)_L$ model can be written as:

$$Y_t = \gamma + \sum_{L=1}^{P} \phi_L Y_{t-PL} + \varepsilon_t, \quad (1)$$

where γ is the process mean, ϕ_L is the coefficient of the SAR model $(-1 < \phi_L < 1)$ and ε_t is the exponential white noise sequence of independent random variables as $\varepsilon_t \sim Exp(\beta)$.

2.2 Control Charts

2.2.1 The Exponentially Weighted Moving Average Control Chart

EWMA is a control chart suitable for moving averages of past and present data; it can detect process mean values less than 1.5σ or lower. The EWMA control chart was initiated by [5]; the statistics of the EWMA control chart are as follows:

$$E_t = \lambda Y_t + (1-\lambda) E_{t-1} \quad (2)$$

where λ is an exponential smoothing parameter $(0 < \lambda \le 1)$, Y_t is a sequence of $SAR(P)_L$ model with exponential white noise.

The UCL and LCL of the EWMA control chart are specified as follows:

$$\mu \pm W_E \sigma \sqrt{\frac{\lambda}{2-\lambda}}, \quad (3)$$

where μ is the target mean, σ is the standard deviation of the process, and W_E is the width of the control limits.

The stopping time of the EWMA control chart (τ_h) is denoted as:

$$\tau_h = \{t \geq 0 : E_t < g \text{ or } E_t > h\}$$

where τ_h is the stopping time, and g and h are LCL and UCL, respectively.

2.2.2 The Modified Exponentially Weighted Moving Average Control Chart

The study, [7], introduced a modified exponential weighted moving average (MEWMA) control chart. The MEWMA control chart enhances the EWMA chart, increasing its efficacy in identifying process variations. The MEWMA control chart can be expressed as follows:

$$M_t = (1-\lambda_1)M_{t-1} + \lambda_1 Y_t + f_1(Y_t - Y_{t-L}) \quad (4)$$

where M_0 is initial values and Y_t is a sequence of SAR(P)$_L$ model, f_1 is suitable constant. The upper and lower control limits of the MEWMA control charts are given by:

$$UCL/LCL = \mu_0 \pm W_M \sigma \sqrt{\frac{\lambda}{2-\lambda} + \frac{2f_1(\lambda + f_1)}{2-\lambda}} \quad (5)$$

where μ_0 and σ represent the mean and standard deviation of the process, respectively, and W_M is the width of the appropriate control limit.
The stopping time of the MEWMA control chart (τ_q) is denoted as:

$$\tau_q = \{t \geq 0 : M_t < p \text{ or } M_t > q\}$$

where τ_q is the stopping time, and p and q are LCL and UCL, respectively.

2.2.3 The Double Modified Exponentially Weighted Moving Average Control Chart

The study, [8], constructed a double MEWMA control chart that is capable of rapidly identifying process changes. The DMEWMA statistics can be defined as follows.

$$DM_t = (1-\lambda_2)DM_{t-1} + \lambda_2 M_t + f_2(M_t - M_{t-1}), \quad (6)$$

where $M_t = (1-\lambda_1)M_{t-1} + \lambda_1 Y_t + f_1(Y_t - Y_{t-L})$ is the MEWMA statistic, λ_1 and λ_2 are exponential smoothing parameters $(0 < \lambda_1, \lambda_2 \leq 1)$, f_1 and f_2 are appropriate constants, and Y_t is a sequence of SAR(P)$_L$ model at t = 1, 2, 3,....

The lower and upper control limits can be expressed as:

$$UCL/LCL = \mu_0 \pm W_{DM} \sigma \sqrt{F_\sigma} \quad (7)$$

where W_{DM} contains appropriate width of control limits and F_σ is a standard deviation with $f = f_1 = f_2$, $\lambda = \lambda_1 = \lambda_2$ and $\theta = (1 - \lambda)^2$ outlined as

$$\begin{aligned} F_\sigma &= (f+\lambda)^4 + 4\lambda^2 (f+\lambda)^2 (f+\lambda-1)^2 \\ &+ \frac{4\lambda^2 (f+\lambda)^2 (f+\lambda-1)^2 (1-\lambda)^2}{1-\theta} \\ &- \frac{4\lambda^3 (f+\lambda)(f+\lambda-1)^3 (1-\lambda)}{(1-\theta)^2} \\ &+ \lambda^4 (f+\lambda-1)^4 \frac{(1+\theta)}{(1-\theta)^3}. \end{aligned}$$

The stopping time of the DMEWMA control chart (τ_b) is denoted as:

$$\tau_b = \{t \geq 0 : M_t < a \text{ or } M_t > b\}$$

where τ_b is the stopping time, and a and b are LCL and UCL, respectively.

3 Methods of Evaluating Average Run Length of Control Chart

3.1 The Exact Solution of ARL on the DMEWMA Control Chart for SAR(P)$_L$ process

The time series data utilizing the SAR(P)$_L$ model with exponential white noise can be expressed as follows.

$$Y_t = \gamma + \sum_{L=1}^{P} \phi_L Y_{t-PL} + \varepsilon_t, \quad (8)$$

where γ is the process mean, ϕ_L is the coefficient of the SAR(P)$_L$ model $(-1 < \phi_L < 1)$ and ε_t is the exponential white noise sequence of independent random variables as $\varepsilon_t \sim Exp(\beta)$.

Therefore, the DMEWMA statistic for a SAR(P)$_L$ model is given by:

$$\begin{aligned} DM_t &= (1-\lambda_2)DM_{t-1} + [(1-\lambda_1)\lambda_2 + (1-\lambda_1) f_2 - f_2]M_{t-1} \\ &+ (\lambda_1\lambda_2 + f_1\lambda_2 + f_2\lambda_1 + f_1 f_2)(\gamma + \varepsilon_t) \\ &+ (\lambda_1\lambda_2 + f_1\lambda_2 + f_2\lambda_1 + f_1 f_2)\sum_{L=1}^{P} \phi_l Y_{t-PL} - (f_1\lambda_2 + f_1 f_2)Y_{t-L}. \end{aligned}$$

Since the interval at DM_1 in the control process is [a, b] to determine the initial value $DM_0 = u$, we can acquire

$$\begin{aligned} \vartheta &= (\lambda_1\lambda_2 + f_1\lambda_2 + f_2\lambda_1 + f_1 f_2)\gamma \\ &+ (\lambda_1\lambda_2 + f_1\lambda_2 + f_2\lambda_1 + f_1 f_2)\sum_{L=1}^{P} \phi_L A_{1-L} - (f_1\lambda_2 + f_1 f_2)Y_0 \\ &+ [(1-\lambda_1)\lambda_2 + (1-\lambda_1) f_2 - f_2]* \\ &[(1-\lambda_1)M_{-L} + \lambda_1 Y_0 + f_1(Y_0 - Y_{-L})], \end{aligned}$$

then $a \leq (1-\lambda_2)u + (\lambda_1\lambda_2 + f_1\lambda_2 + f_2\lambda_1 + f_1 f_2)\varepsilon_1 + \vartheta \leq b.$

After that, the control limit is transferred to the exponential residual term ε_t, as follows:

$$\frac{a-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)} \le \varepsilon_1 \le \frac{b-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}$$

such that

$$P(LCL \le \varepsilon_1 \le UCL) = \int_{\frac{b-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}^{\frac{a-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}} f(\varepsilon_1)d\varepsilon_1.$$

The ARL can be resolved utilizing the Fredholm integral equation of the second kind, the ARL denoted $\kappa(u)$ can be expressed as

$$\kappa(u) = 1 + \int_{\frac{a-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}^{\frac{b-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+c_1\lambda_2+c_2\lambda_1+c_1c_2)}} Y\begin{bmatrix}(1-\lambda)u+\vartheta \\ +(\lambda_1\lambda_2+f_1\lambda_2 \\ +f_2\lambda_1+f_1f_2)x\end{bmatrix} f(x)dx$$

$$\kappa(u) = 1 + \frac{1}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)} * \int_a^b \kappa(z) f\left[\frac{z-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}\right]dz. \quad (9)$$

The explicit formula for the ARL for the DMEWMA statistic and SAR(P)$_L$ model can be obtained by resolving the integral equation in Eq. (9) using an exponential function as demonstrated:

$$\kappa(u) = 1 + \frac{e^{\frac{(1-\lambda_2)u+\vartheta}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)} * \int_a^b e^{\frac{-z}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\kappa(z)dz.$$

If $M = \int_a^b e^{\frac{-z}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\kappa(z)dz$, then

$$\kappa(u) = 1 + \frac{e^{\frac{(1-\lambda_2)u+\vartheta}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)} * M.$$

Let $P = \beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)$, then

$$\kappa(u) = 1 + \frac{e^{\frac{(1-\lambda_2)u+\vartheta}{P}}}{P} * M.$$

Moreover, M can be solved as follows:

$$M = \int_a^b e^{\frac{-z}{P}}\left(1 + \frac{e^{\frac{(1-\lambda_2)z+\vartheta}{P}}}{P}\cdot M\right)dz$$

$$M = -P(e^{\frac{-b}{P}} - e^{\frac{-a}{P}}) - \frac{e^{\frac{\vartheta}{P}}}{\lambda_2}(e^{\frac{-b\lambda_2}{P}} - e^{\frac{-a\lambda_2}{\beta\gamma}})M$$

Afterwards,

$$\kappa(u) = 1 + \frac{e^{\frac{(1-\lambda_2)u+\vartheta}{P}}}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)} * \left(\frac{-P(e^{\frac{-b}{P}} - e^{\frac{-a}{P}})}{1 + \frac{e^{\frac{\vartheta}{P}}}{\lambda_2}(e^{\frac{-b\lambda_2}{P}} - e^{\frac{-a\lambda_2}{P}})}\right)$$

$$\kappa(u) = 1 - \frac{\lambda_2 e^{\frac{(1-\lambda_2)u}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\left(e^{\frac{-b}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}} - e^{\frac{-a}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\right)}{\lambda_2 e^{\frac{-\vartheta}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}} + e^{\frac{-b\lambda_2}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}} - e^{\frac{-a\lambda_2}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}}. \quad (10)$$

For One-sided DMEWMA, in Eq. 10, a will equal 0; thus the explicit formula for the ARL is as follows:

$$\kappa(u) = 1 - \frac{\lambda_2 e^{\frac{(1-\lambda_2)u}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\left(e^{\frac{-b}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}} - 1\right)}{\lambda_2 e^{\frac{-\vartheta}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}} + e^{\frac{-b\lambda_2}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}} - 1}. \quad (11)$$

Consequently, $\kappa(u)$ in Eq. (10) and Eq. (11) represents the explicit formula for the ARL of the DMEWMA control chart for SAR(P)$_L$ model.

3.1.1 The Existence and Uniqueness of Explicit Formulas

We illustrate that the integral equation given in Eq. (8) has a solution and that it is unique. We first outline:

$$T(\kappa(u)) = 1 + \frac{e^{\frac{(1-\lambda_2)u+\vartheta}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)} * \int_a^b e^{\frac{-z}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\kappa(z)dz. \quad (12)$$

Theorem 1. (Banach's fixed-point theorem)
Let $H[a,b]$ be the set comprising all of the continuous functions on complete metric (X,d), and presume that $T: X \to X$ is a contraction mapping with contraction constant $0 \le s < 1$; i.e.,

$$\|T(\kappa_1)-T(\kappa_2)\| \le s\|\kappa_1-\kappa_2\| \ \forall \kappa_1,\kappa_2 \in X$$

Subsequently, $\vartheta(.) \in X$ is unique at $T(\kappa(u)) = \kappa(u)$; i.e., it has a unique fixed point in X.

Proof: To demonstrate that *T*, as described in Eq. (11), constitutes a contraction mapping for $\kappa_1,\kappa_2 \in H[a,b]$, we employ the inequality $\|T(\kappa_1)-T(\kappa_2)\| \le s\|\kappa_1-\kappa_2\|$ $\forall \kappa_1,\kappa_2 \in C(a,b)$ with $0 \le s < 1$.

$$\|T(\kappa_1)-T(\kappa_2)\|_\infty = \sup_{u\in[a,b]}\left|F(u)\int_g^h(\kappa_1(z)-\kappa_2(z))e^{\frac{-z}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}dz\right|$$

$$\le \sup_{u\in[a,b]}\left|\|\kappa_1-\kappa_2\|_\infty F(u)\left(e^{\frac{-a}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}-e^{\frac{-b}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\right)\right|$$

$$=\|\kappa_1-\kappa_2\|_\infty\left|e^{\frac{-a}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}-e^{\frac{-b}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\right|\sup_{u\in[a,b]}|F(u)|$$

$\le s\|\kappa_1-\kappa_2\|_\infty$, where

$$s=\left|e^{\frac{-a}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}-e^{\frac{-b}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}\right|\sup_{u\in[a,b]}|F(u)|$$

and $F(u)=\dfrac{e^{\frac{(1-\lambda_2)u+\vartheta}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}}}{\beta(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}$; $0 \le s < 1$.

Thus, as confirmed by Banach's fixed-point theorem, the solution is both existent and unique.

3.2 The Approximating ARL on the DMEWMA Control Chart for SAR(P)$_L$ utilizing NIE

The estimated ARL using the NIE method derived by using the composite midpoint quadrature rule denoted $\kappa(u)$ is a well-known technique that can be used to verify the ARL via the explicit formula. This rule gives ARL values close to other techniques and the lowest CPU time. $\kappa(u)$ for a SAR(P)$_L$ model on the DMEWMA control chart can be determined via the *k* linear equation system on the interval [*a*,*b*], where the length of *k* is equally divided into intervals, i.e., $h_j=\dfrac{b-a}{k}$ with the middle point of the j^{th} interval $z_j=(j-\frac{1}{2})h_j+a.$ From Eq. (9), $\kappa_N(u)$ can be defined as:

$$\kappa_N(u)\approx 1+\frac{1}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}*\sum_{j=1}^k h_j\cdot\kappa(z_j)\,f\left[\frac{z_j-(1-\lambda_2)u-\vartheta}{(\lambda_1\lambda_2+f_1\lambda_2+f_2\lambda_1+f_1f_2)}\right]$$

This study contrasts the outcomes for ARL$_0$ and ARL$_1$ utilizing explicit formulas and the NIE procedure of SAR(P)$_L$ beside the DMEWMA control chart. The precision of the ARL has been compared with the accuracy percentage, which can possibly be obtained from:

$$\%Acc = 100 - \left|\frac{\kappa(u)-\kappa_N(u)}{\kappa(u)}\right|\times 100\%.$$

The performance of control charts is further assessed using the standard deviation of the run length (SDRL) and the median run length (MRL), [21]. For a controlled process, the SDRL and MRL values are computed using the following formulas:

$$ARL_0=\frac{1}{\rho_0}, SDRL_0=\sqrt{\frac{1-\rho_0}{\rho_0^{\,2}}}, MRL_0=\frac{\log(0.5)}{\log(1-\rho_0)}, \quad (13)$$

where π_0 represents a type I error. In this study, ARL_0 was established at 370 throughout this investigation, and it can be determined using $SDRL_0$, via Eq. (12). To calculate $SDRL_1, MRL_1$ for an out-of-control situation, substitute ρ_0 with ρ_1 where ρ_1 represents type II error. The control chart most effective at monitoring changes in the process mean will display the lowest ARL_1, $SDRL_1$, and MRL_1 values. In addition, the criteria were used to judge how well control charts worked as follows: expected average run length (EARL), expected standard deviation run length (ESDRL), and expected median run length (EMRL), as explained in the study, [22].

The EARL can be expressed mathematically as:

$$EARL=\frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} ARL(\delta)$$

The ESDRL is characterised via:

$$ESDRL=\frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} SDRL(\delta)$$

The EMRL is delineated as:

$$EMRL=\frac{1}{\Delta}\sum_{\delta=\delta_{\min}}^{\delta_{\max}} MRL(\delta)$$

where the $\delta_{\min}$ and represents the lower and upper bounds of shift parameter (δ), $ARL(\delta)$ is the ARL_1 value for a specific shift and Δ denotes the total number of increments between $\delta_{\min}$ and $\delta_{\max}$. $ESDRL(\delta), EMRL(\delta)$ can be computed in a manner comparable to $EARL(\delta)$.

4 Numerical Results

Table 1 (Appendix) shows the results of ARL values on a one-sided DMEWMA control chart from the explicit formula and the NIE method at the change levels of 0.005, 0.007,…,1.0 for the $SAR(2)_{12}$ model at different levels of ϕ_i values and giving λ_1 equal to 0.05 and λ_2 equal to 0.05, 0.10 and 0.20. The results of the research found that the ARL values obtained from both methods were very close. It was found that the processing time of the explicit formula method was less than 0.001 seconds. In addition, when considering the %ACC values, it was found that both methods were very similar, with values equal to 100% at all levels of change. Table 2 (Appendix). The $SAR(3)_4$ model for a two-sided DMEWMA control chart was studied. The value of ϕ_3 was set to 0.3 and the lower control limit was set to 0.1. Several scenarios of ϕ_1, ϕ_2 and λ_2 were determined. The results of ARL analysis were similar to those in Table 1 (Appendix). The values of ARL obtained by the explicit formula and NIE method were very close, as confirmed by the %ACC values.

After finding that the explicit formula of the DMEWMA control chart is accurate, the DMEWMA control chart is compared with the MEWMA and EWMA control charts, and the results are shown in Table 3 (Appendix). Both one-sided and two-sided were considered. The studied models were set to $SAR(2)_4$, λ_1 and λ_2 equal to 0.1, 0.2, respectively. DMEWMA control chart has set f_1, f_2 values in many situations, while the MEWMA control chart sets f_1 equal 0.5, 2.5 and 5. The results showed that both one-sided and two-sided results gave the same results. The DMEWMA control chart has the fastest efficiency in detecting changes at all levels of change. If we consider only the DMEWMA control chart, we will find that f_1, f_2 equal to 5 can detect process changes the best because they give the lowest ARL values. In addition, if we consider the criteria of EARL, ESDRL and EMRL values, we can confirm that the DMEWMA control chart with f_1, f_2 equal to 5 is the most efficient in detecting changes because it gives the lowest values of all three values. To validate the findings in Table 4 (Appendix), a $SAR(3)_{12}$ model study was performed to compare the efficacy of three control charts. The analysis revealed that DMEWMA control chart exhibited superior efficiency in detecting mean alterations at f_1, f_2 levels of 5, demonstrating the lowest values of ARL, SDRL, and EMRL. Additionally, the values of EARL, ESDRL, and EMRL were also identified as the lowest. The results agree with the conclusions described in Table 3 (Appendix). The DMEWMA control chart indicates enhanced performance relative to the MEWMA and EWMA control charts.

4.1 Application

The monthly production index for Biofuel Gasohol 91 from January 2021 to December 2023 is utilized to assess the effectiveness of the explicit formulas for the ARL on the DMEWMA control chart, in comparison to the MEWMA and EWMA control charts as shown in Table 5 (Appendix). The parameters λ_1 and λ_2 are assigned the following values: 0.2 and 0.3, respectively. The subsequent coefficient parameters are obtained for $SAR(1)_{12}$, based on the model estimation executed by maximum likelihood estimation: $\phi_1 = 0.988$. The exponential parameter is set at 52.4797, as seen in Table 6 (Appendix). Applying the parameters of this forecasting model, the subsequent can be illustrated as $\hat{Y}_t = 0.988Y_{t-12}$. The analysis of one-sided and two-sided control charts revealed that the DMEWMA control chart was the most efficient, as it consistently demonstrated the lowest ARL, SDRL, and MRL values across all levels of variation. The findings corresponded to the results in Appendix in Table 3 and Table 4, indicating that the DMEWMA control chart exhibited optimal efficiency when the F1 and F2 values were 5, corresponding to the lowest EARL, ESDRL, and EMRL values. This result is clearly illustrated in Figure 1 (Appendix), which compares one-sided and two-sided control charts. In addition, when calculating the statistical value of each control chart according to equations 2, 3, and 5 and setting the control limit of the chart to have an ARL_0 value equal to 370, it was found that the DMEWMA control chart was able to detect the change the fastest, with the first observation that went out of control limit at the first sequence. In contrast, the MEWMA control chart had the first observation that went out of the control limit at the third sequence, and the EWMA control chart had the first observation that went out of the control limit at the second sequence, as shown in Figure 2 (Appendix).

5 Conclusion

This study applied the explicit ARL calculation formula for the $SAR(P)_L$ process on the DMEWMA control chart. The formula demonstrated excellent accuracy and efficiency by greatly reducing computation time. A comparison of ARL results between the explicit formula and the numerical integral equation (NIE) method was conducted using the percentage of accuracy (%Acc) criterion. The results showed no significant differences between the two methods, confirming the reliability of the explicit formula. In addition, the explicit formula significantly reduces computation time compared to the NIE method, as demonstrated by the results. When evaluating the performance of DMEWMA, MEWMA, and EWMA control charts in detecting process changes, the findings reveal that the DMEWMA control chart outperforms other types. This result is evidenced by its lowest values for EARL, ESDRL, and EMRL, making it the most efficient in identifying shifts in the process mean.

The study confirms that the simulation validation results are consistent with real data applications, supporting the conclusion that the DMEWMA control chart performs best. Future research could focus on developing an explicit formula for ARL values in the DMEWMA control chart to support new types of control charts or explore its application in other advanced models.

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Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work the authors used QuillBot in order to modify a few phrases in the research work citation in Introduction section to enhance academic quality. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Saowanit Sukparungsee has implemented the methodology and simulation.
- Yupaporn Areepong has organized the writing-original draft, conceptualization, and validation

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Conflicts of Interest

The authors declare no conflict of interest.

APPENDIX

Table 1. ARL comparison using explicit formulas and the NIE method for $SAR(2)_{12}$ process on one-sided DMEWMA control chart for different choices of ϕ with $f_1 = 0.5, f_2 = 1.5$, $\lambda_1 = 0.05$ $\gamma = 0.05, ARL_0 = 370$

λ_2	Coefficients of process			Methods	Shift size (δ)						
	ϕ_1	ϕ_2	b		0.005	0.007	0.01	0.05	0.07	0.1	1.0
0.05				Explicit	272.855	246.573	215.273	77.4940	57.8516	41.4407	3.88897
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
	0.1	0.1	1.78615	NIE	272.855	246.573	215.273	77.4940	57.8516	41.4407	3.88897
				CPU_{NIE}	(5.188)	(5.265)	(4.625)	(5.000)	(4.953)	(4.500)	(4.578)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	343.356	333.114	318.247	176.700	136.244	96.7373	5.13303
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
		-0.1	2.2094	NIE	343.355	333.113	318.247	176.700	136.244	96.7373	5.13303
				CPU_{NIE}	(5.188)	(5.062)	(5.078)	(5.172)	(5.110)	(5.203)	(5.156)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	251.907	223.181	190.476	63.2302	46.9727	33.6704	3.51872
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
	0.2	0.1	1.60746	NIE	251.907	223.181	190.476	63.2302	46.9727	33.6704	3.51872
				CPU_{NIE}	(5.078)	(5.172)	(4.328)	(5.172)	(5.125)	(5.172)	(5.375)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	301.719	280.424	253.277	104.667	78.8303	56.3469	4.39368
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
		-0.1	1.98588	NIE	301.719	280.424	253.277	104.667	78.8303	56.3469	4.39368
				CPU_{NIE}	(4.907)	(4.938)	(4.703)	(4.859)	(4.687)	(4.829)	(4.890)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.10				Explicit	268.385	241.547	209.881	74.1766	55.2950	39.5988	3.81202
			1.8359	CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
	0.1	0.1		NIE	268.385	241.547	209.881	74.1765	55.2950	39.5987	3.81202
				CPU_{NIE}	(4.969)	(5.078)	(4.312)	(4.594)	(4.828)	(5.109)	(4.844)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	346.713	337.289	323.433	183.283	141.346	100.044	5.14335
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
		-0.1	2.30114	NIE	346.712	337.289	323.432	183.283	141.346	100.044	5.14335
				CPU_{NIE}	(4.562)	(4.938)	(4.390)	(4.313)	(4.359)	(4.781)	(4.203)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	246.289	217.061	184.168	59.9298	44.4651	31.8735	3.43197
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
	0.2	0.1	1.64332	NIE	246.289	217.060	184.168	59.9297	44.4651	31.8735	3.43197
				CPU_{NIE}	(4.500)	(4.016)	(4.875)	(4.422)	(4.531)	(4.422)	(3.844)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	299.255	277.620	250.161	102.190	76.8541	54.8921	4.34147
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
		-0.1	2.0538	NIE	299.254	277.620	250.161	102.190	76.8541	54.8921	4.34147
				CPU_{NIE}	(4.750)	(4.625)	(4.594)	(4.578)	(4.079)	(4.641)	(4.656)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.20				Explicit	261.627	233.893	201.663	69.2646	51.5299	36.8983	3.70378
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
	0.1	0.1	1.94785	NIE	261.627	233.893	201.663	69.2645	51.5298	36.8982	3.70378
				CPU_{NIE}	(4.813)	(4.375)	(4.875)	(4.922)	(4.890)	(4.297)	(4.844)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	356.627	350.665	341.319	213.294	165.712	116.547	5.28672
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
		-0.1	2.5174	NIE	356.625	350.663	341.318	213.293	165.712	116.547	5.28672
				CPU_{NIE}	(4.375)	(4.578)	(5.157)	(4.953)	(5.047)	(4.875)	(5.062)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	246.289	217.061	184.168	59.9298	44.4651	31.8735	3.43197
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
	0.2	0.1	1.72295	NIE	246.289	217.060	184.168	59.9297	44.4651	31.8735	3.43197
				CPU_{NIE}	(4.906)	(5.203)	(4.844)	(4.953)	(5.156)	(5.203)	(5.313)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
				Explicit	299.255	277.620	250.161	102.190	76.8541	54.8921	4.34147
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
		-0.1	2.20973	NIE	299.254	277.620	250.161	102.190	76.8541	54.8921	4.34147
				CPU_{NIE}	(4.953)	(5.156)	(5.031)	(5.157)	(5.140)	(5.032)	(5.406)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Note: *The numerical results in parentheses are computational times in seconds*

Table 2. ARL comparison using explicit formulas and the NIE method for $SAR(3)_4$ process on two-sided DMEWMA control chart for different choices of ϕ with $a = 0.1$, $\lambda_1 = 0.05$, $\gamma = 0.10, \phi_3 = 0.3, ARL_0 = 370$

λ_2	Coefficients of process			Methods	Shift size (δ)						
	ϕ_1	ϕ_2	b		0.005	0.007	0.01	0.05	0.07	0.1	0.5
0.05	0.1	0.1	1.346	Explicit	209.284	178.207	145.723	42.3199	31.1987	22.3712	4.9266
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	209.284	178.207	145.723	42.3199	31.1987	22.3712	4.9266
				CPU_{NIE}	(5.187)	(4.985)	(4.593)	(4.312)	(4.188)	(4.688)	(4.687)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		-0.1	1.63534	Explicit	235.381	205.283	172.178	53.9677	39.9538	28.644	5.9621
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	235.381	205.283	172.178	53.9677	39.9538	28.644	5.9621
				CPU_{NIE}	(4.937)	(5.000)	(4.609)	(3.843)	(4.875)	(4.266)	(4.813)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	0.2	0.1	1.22317	Explicit	199.67	168.602	136.687	38.722	28.5006	20.4282	4.56654
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	199.67	168.602	136.687	38.722	28.5006	20.4282	4.56654
				CPU_{NIE}	(4.750)	(4.516)	(5.109)	(4.671)	(4.516)	(4.859)	(4.688)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		-0.1	1.48281	Explicit	220.766	189.977	157.073	47.1252	34.8076	24.963	5.3765
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	220.766	189.977	157.073	47.1252	34.8076	24.963	5.3765
				CPU_{NIE}	(4.969)	(5.015)	(4.469)	(4.234)	(4.516)	(4.656)	(4.719)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.10	0.1	0.1	1.36769	Explicit	203.357	172.282	140.146	40.1323	29.5754	21.2211	4.75224
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	203.357	172.282	140.146	40.1323	29.5754	21.2211	4.75224
				CPU_{NIE}	(4.594)	(5.016)	(5.234)	(4.422)	(4.782)	(4.828)	(4.735)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		-0.1	1.6756	Explicit	230.893	200.549	167.468	51.7835	38.3126	27.4753	5.79687
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	230.893	200.549	167.468	51.7835	38.3126	27.4753	5.79687
				CPU_{NIE}	(5.172)	(5.094)	(4.687)	(4.750)	(5.172)	(5.203)	(4.906)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	0.2	0.1	1.23864	Explicit	193.604	162.575	131.065	36.5949	26.9268	19.3146	4.3949
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	193.604	162.575	131.065	36.5949	26.9268	19.3146	4.3949
				CPU_{NIE}	(4.828)	(5.125)	(4.812)	(5.484)	(4.984)	(5.016)	(4.953)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		-0.1	1.5126	Explicit	215.653	184.679	151.916	44.9166	33.158	23.7905	5.20313
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	215.653	184.679	151.916	44.9166	33.158	23.7905	5.20313
				CPU_{NIE}	(5.344)	(4.454)	(4.734)	(5.063)	(5.203)	(5.281)	(4.906)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.20	0.1	0.1	1.4153	Explicit	193.337	162.289	130.794	36.5635	26.9344	19.353	4.46761
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	193.337	162.289	130.794	36.5635	26.9344	1(9.353	4.46761
				CPU_{NIE}	(4.938)	(5.093)	(5.219)	(5.125)	(4.734)	(4.703)	(4.625)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		-0.1	1.76501	Explicit	223.312	192.687	159.772	48.3875	35.7748	25.6761	5.54673
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	223.312	192.687	159.772	48.3875	35.7748	25.6761	5.54673
				CPU_{NIE}	(5.047)	(5.063)	(5.093)	(4.813)	(4.672)	(4.312)	(5.313)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	0.2	0.1	1.27271	Explicit	182.613	151.874	121.275	33.077	24.3349	17.4853	4.11128
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	182.613	151.874	121.275	33.077	24.3349	17.4853	4.11128
				CPU_{NIE}	(5.000)	(5.125)	(5.390)	(4.625)	(5.063)	(4.797)	(4.906)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		-0.1	1.57815	Explicit	206.565	175.451	143.108	41.3435	30.5028	21.9099	4.92614
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	206.565	175.451	143.108	41.3435	30.5028	21.9099	4.92613
				CPU_{NIE}	(5.141)	(5.078)	(5.594)	(5.234)	(4.453)	(4.594)	(4.578)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Note: *The numerical results in parentheses are computational times in seconds*

Table 3. ARL comparison for the SAR(2)$_4$ process on one-sided and two-sided EWMA, MEWMA, and DMEWMA control charts with $\lambda_1 = 0.1, \lambda_2 = 0.2, \gamma = 0.01, \phi_1 = 0.15, \phi_2 = 0.2$ $\beta = 1$, and $ARL_0 = 370$

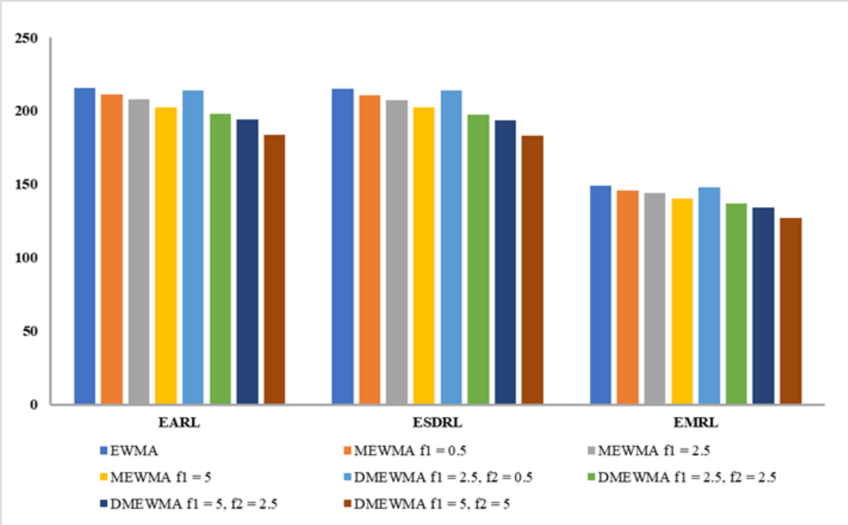
a	δ	Control Chart	DMEWMA				MEWMA			EWMA
			f_1= 2.5, f_2=0.5	f_1= 2.5, f_2= 2.5	f_1= 5, f_2= 2.5	f_1= 5, f_2= 5	f_1= 0.5	f_1 = 2.5	f_1 = 5	f_1 = 0
0			b = 2.59222	b= 9.72835	b = 19.185	b = 24.444	q= 1.0428	q = 5.22835	q= 10.47575	h= 0.00309
	0.000	ARL_0	370.424	370.288	370.483	370.800	370.861	370.982	370.265	370.747
		$SDRL_0$	369.924	369.788	369.983	370.300	370.361	370.482	369.765	370.247
		MRL_0	256.412	256.317	256.453	256.672	256.715	256.798	256.301	256.636
	0.001	ARL_1	296.038	272.671	269.140	**240.689**	345.972	318.311	312.663	366.923
		$SDRL_1$	295.538	272.171	268.640	**240.188**	345.472	317.811	312.163	366.423
		MRL_1	204.851	188.654	186.207	**166.486**	239.463	220.290	216.375	253.985
	0.003	ARL_1	211.313	178.756	174.203	**141.736**	304.914	247.897	238.468	359.413
		$SDRL_1$	210.812	178.255	173.702	**141.235**	304.414	247.396	237.967	358.913
		MRL_1	146.124	123.557	120.401	**97.897**	211.004	171.482	164.947	248.779
	0.005	ARL_1	164.375	133.122	128.950	**100.643**	272.443	202.978	192.735	352.085
		$SDRL_1$	163.874	132.621	128.449	**100.142**	271.943	202.477	192.234	351.585
		MRL_1	113.589	91.9261	89.0343	**69.4133**	188.496	140.347	133.247	243.700
	0.01	ARL_1	105.835	81.5207	78.4543	**58.6435**	214.828	139.668	130.282	334.528
		$SDRL_1$	105.334	81.0192	77.9527	**58.1414**	214.327	139.167	129.781	334.028
		MRL_1	73.0121	56.1586	54.0331	**40.3010**	148.561	96.4635	89.9576	231.530
	0.03	ARL_1	44.0156	32.5275	31.1584	**22.5482**	115.040	62.0924	56.7874	273.946
		$SDRL_1$	43.5127	32.0236	30.6543	**22.0425**	114.539	61.5904	56.2852	273.446
		MRL_1	30.1614	22.1980	21.2489	**15.2800**	79.3926	42.6917	39.0144	189.538
	0.05	ARL_1	28.0189	20.6622	19.7973	**14.3018**	77.6226	39.9166	36.3643	226.001
		$SDRL_1$	27.5144	20.1560	19.2908	**13.7927**	77.1210	39.4134	35.8608	225.500
		MRL_1	19.0725	13.9725	13.3729	**9.56249**	53.4566	27.3200	24.8576	156.305
	0.10	ARL_1	14.9609	11.1976	10.7596	**7.85199**	41.7359	21.1428	19.2714	143.945
		$SDRL_1$	14.4523	10.6859	10.2474	**7.33497**	41.2329	20.6367	18.7647	143.444
		MRL_1	10.0195	7.40961	7.10578	**5.08814**	28.5811	14.3057	13.0083	99.4281
	0.3	ARL_1	5.71547	4.58166	4.45107	**3.40431**	13.4467	7.61684	7.06126	33.4790
		$SDRL_1$	5.19145	4.05092	3.91931	**2.86095**	12.9370	7.09925	6.54218	32.9752
		MRL_1	3.60399	2.81498	2.72399	**1.99307**	8.96951	4.92489	4.53910	22.8575
	0.50	ARL_1	3.82425	3.21598	3.14687	**2.48952**	7.73382	4.89544	4.60809	11.7377
		$SDRL_1$	3.28643	2.66956	2.59922	**1.92567**	7.21652	4.36691	4.07755	11.2266
		MRL_1	2.28671	1.86111	1.81264	**1.34949**	5.00611	3.03350	2.83339	7.78424
	EARL		87.65034	74.14706	72.21997	**59.40914**	104.74300	100.10492	104.74300	210.47411
	ESDRL		87.13558	73.63216	71.70157	**58.88462**	104.23178	99.59299	104.23178	209.96648
	EMRL		60.40119	51.04136	49.70398	**40.82136**	72.25055	69.03535	72.25055	145.53943
0.1			b= 2.0749	b = 6.79335	b = 19.3174	b = 24.5644	q = 1.16303	q = 5.35105	q = 10.5989	h= 0.10806
	0.000	ARL_0	370.458	370.311	370.300	370.616	370.007	370.577	370.016	370.317
		$SDRL_0$	369.958	369.811	369.800	370.116	369.507	370.077	369.516	369.817
		MRL_0	256.435	256.333	256.326	256.545	256.123	256.518	256.129	256.337
	0.001	ARL_1	277.288	248.253	268.751	**240.402**	342.461	316.981	311.945	366.840
		$SDRL_1$	276.788	247.752	268.251	**239.901**	341.961	316.481	311.445	366.340
		MRL_1	191.855	171.729	185.937	**166.287**	237.029	219.368	215.877	253.927
	0.003	ARL_1	184.647	149.894	173.798	**141.493**	297.970	245.850	237.426	360.005
		$SDRL_1$	184.146	149.393	173.297	**140.992**	297.470	245.349	236.925	359.505
		MRL_1	127.641	103.552	120.121	**97.7285**	206.190	170.063	164.224	249.190
	0.005	ARL_1	138.525	107.545	128.598	**100.449**	263.596	200.780	191.648	353.323
		$SDRL_1$	138.024	107.044	128.097	**99.9477**	263.096	200.279	191.147	352.823
		MRL_1	95.6712	74.1974	88.7903	**69.2788**	182.364	138.823	132.493	244.558
	0.01	ARL_1	85.4693	63.3050	78.2059	**58.5194**	204.297	137.661	129.326	337.260
		$SDRL_1$	84.9678	62.8030	77.7043	**58.0172**	203.796	137.160	128.825	336.760
		MRL_1	58.8956	43.5322	53.8609	**40.2150**	141.261	95.0723	89.2949	233.424
	0.03	ARL_1	34.1887	24.4883	31.0513	**22.4989**	106.418	60.9534	56.2667	281.212
		$SDRL_1$	33.6850	23.9831	30.5472	**21.9932**	105.917	60.4513	55.7645	280.712
		MRL_1	23.3495	16.6250	21.1747	**15.2458**	73.4162	41.9021	38.6535	194.575
	0.05	ARL_1	21.6368	15.5149	19.7303	**14.2716**	71.1949	39.1530	36.0183	236.053
		$SDRL_1$	21.1309	15.0066	19.2238	**13.7625**	70.6931	38.6498	35.5148	235.552
		MRL_1	14.6482	10.4037	13.3264	**9.54155**	49.0012	26.7907	24.6178	163.273
	0.10	ARL_1	11.5930	8.46983	10.7263	**7.83718**	38.1145	20.7409	19.0897	156.563
		$SDRL_1$	11.0817	7.95413	10.2141	**7.32012**	37.6112	20.2347	18.5830	156.062
		MRL_1	7.68387	5.51701	7.08268	**5.07786**	26.0708	14.0271	12.8823	108.174
	0.3	ARL_1	4.57951	3.60176	10.7263	**3.40016**	12.4406	7.49840	7.00689	41.3585
		$SDRL_1$	4.04875	3.06120	10.2141	**2.85673**	11.9301	6.98052	6.48765	40.8554
		MRL_1	2.81348	2.13122	7.08268	**1.99016**	8.27175	4.84266	4.50134	28.3195
	0.50	ARL_1	3.14950	2.60220	3.14220	**2.48745**	7.26071	4.83456	4.57974	15.7128
		$SDRL_1$	2.60189	2.04187	2.59446	**1.92353**	6.74220	4.30563	4.04898	15.2046
		MRL_1	1.81448	1.42923	1.80936	**1.34802**	4.67761	2.99111	2.81364	10.5409
	EARL		76.31516	62.55091	72.68671	**59.31421**	134.74376	103.73425	99.61048	215.20331
	ESDRL		75.79673	62.02773	72.17023	**58.78967**	134.23610	103.22289	99.09847	214.69794
	EMRL		52.54254	42.99968	50.02839	**40.75555**	93.04714	71.55128	68.69260	148.81851

Table 4. ARL comparison for the SAR(3)$_{12}$ process on one-sided and two-sided EWMA, MEWMA, and DMEWMA control charts with $\lambda_1 = 0.1, \lambda_2 = 0.2,\ \gamma = 0.05,\ \phi_1 = 0.15, \phi_2 = 0.2, \phi_3 = 0.25\ \beta = 1$, and $ARL_0 = 370$

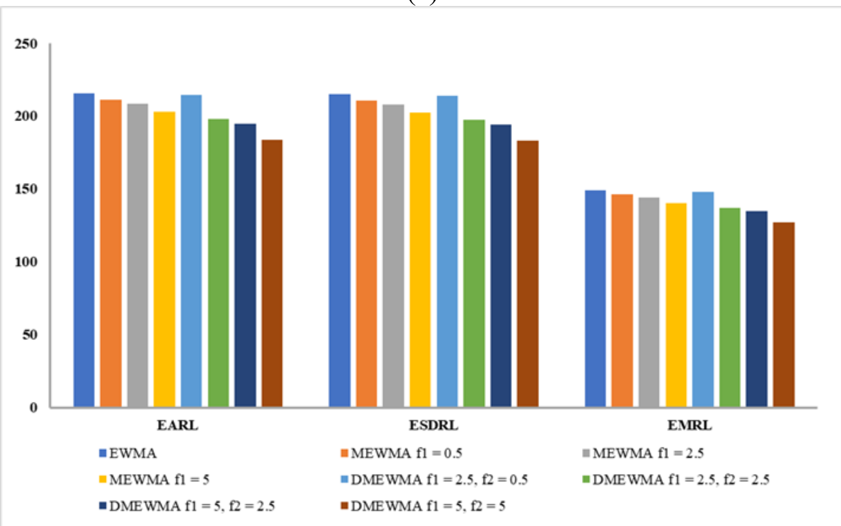
a	δ	Control Chart	DMEWMA				MEWMA			EWMA
			f_1= 2.5, f_2=0.5	f_1= 2.5, f_2= 2.5	f_1= 5, f_2= 2.5	f_1= 5, f_2= 5	f_1= 0.5	f_1 = 2.5	f_1 = 5	f_1 = 0
			b= 1.41832	*b*= 4.8684	*b*= 13.8258	*b*= 17.85	*q*= 0.76189	*q* = 3.80851	*q* = 7.62681	*h*= 0.0023
	0.000	ARL_0	370.082	370.678	370.076	370.155	370.893	370.141	370.047	370.173
		$SDRL_0$	369.582	370.178	369.576	369.655	370.393	369.641	369.547	369.673
		MRL_0	256.175	256.588	256.170	256.225	256.737	256.216	256.150	256.238
	0.001	ARL_1	265.212	234.398	247.315	**225.656**	335.686	288.046	278.338	366.242
		$SDRL_1$	264.712	233.898	246.815	**225.155**	335.186	287.546	277.838	365.742
		MRL_1	183.484	162.126	171.079	**156.066**	232.333	199.312	192.582	253.513
	0.003	ARL_1	169.437	135.359	148.964	**127.033**	282.018	199.681	186.295	358.528
		$SDRL_1$	168.936	134.858	148.463	**126.532**	281.518	199.180	185.794	358.028
		MRL_1	117.098	93.4767	102.907	**87.7055**	195.133	138.062	128.783	248.166
	0.005	ARL_1	124.598	95.3397	106.773	**88.5995**	243.040	152.919	140.148	351.005
		$SDRL_1$	124.097	94.8384	106.272	**88.0981**	242.540	152.418	139.647	350.505
		MRL_1	86.0177	65.7373	73.6623	**61.0653**	168.116	105.648	96.7962	242.951
	0.01	ARL_1	75.1696	55.0865	62.8002	**50.7274**	180.377	96.6485	86.7976	333.001
		$SDRL_1$	74.6679	54.5842	62.2982	**50.2249**	179.876	96.1472	86.2962	332.501
0		MRL_1	51.7563	37.8354	43.1823	**34.8138**	124.681	66.6445	59.8163	230.472
	0.03	ARL_1	29.4476	21.0056	24.3004	**19.2606**	87.9709	39.5591	34.9557	271.101
		$SDRL_1$	28.9433	20.4995	23.7952	**18.7539**	87.4695	39.0559	34.4521	270.601
		MRL_1	20.0630	14.2106	16.4948	**13.0008**	60.6300	27.0722	23.8812	187.566
	0.05	ARL_1	18.5504	13.2861	15.4110	**12.2073**	57.6383	25.1577	22.2226	222.401
		$SDRL_1$	18.0435	12.7763	14.9026	**11.6966**	57.1361	24.6526	21.7168	221.900
		MRL_1	12.5084	8.85813	10.3316	**8.10995**	39.6042	17.0891	15.0543	153.810
	0.10	ARL_1	9.91479	7.26787	8.43392	**6.72634**	30.3858	13.5076	12.0168	139.818
		$SDRL_1$	9.40150	6.74938	7.91815	**6.20623**	29.8816	12.9980	11.5059	139.317
		MRL_1	6.51970	4.68258	5.49209	**4.30648**	20.7133	9.01174	7.97782	96.5675
	0.3	ARL_1	9.91479	3.13729	3.61055	**2.96980**	9.94976	5.29930	4.86183	31.2409
		$SDRL_1$	9.40150	2.58946	3.07010	**2.41866**	9.43652	4.77318	4.33308	30.7368
		MRL_1	6.51970	1.80591	2.13737	**1.68829**	6.54396	3.31455	3.01010	21.3061
	0.50	ARL_1	2.73675	2.29656	2.61755	**2.20324**	5.87720	3.61333	3.38566	10.6975
		$SDRL_1$	2.18015	1.72558	2.05767	**1.62820**	5.35390	3.07292	2.84201	10.1852
		MRL_1	1.52422	1.21244	1.44008	**1.14587**	3.71642	2.13931	1.98001	7.06270
	EARL		70.68256	56.88357	62.20773	**53.70018**	123.60124	82.67669	77.12737	208.64820
	ESDRL		70.16309	56.35641	61.68472	**53.17147**	123.09169	82.16099	76.61048	208.13970
	EMRL		48.63783	39.06954	42.76189	**36.86226**	85.32286	56.95328	53.10624	144.27341
			b = 1.53552	*b*= 4.98343	*b* = 13.9483	*b*= 17.96454	*q* = 0.87644	*q* = 3.92464	*q* = 7.74325	*h*= 0.10598
	0.000	ARL_0	370.067	370.066	370.654	370.079	370.934	370.120	370.439	370.658
		$SDRL_0$	369.567	369.566	370.154	369.579	370.434	369.620	369.939	370.158
		MRL_0	256.164	256.164	256.571	256.172	256.765	256.201	256.422	256.574
	0.001	ARL_1	262.386	233.308	247.188	**225.398**	331.328	286.120	277.514	367.057
		$SDRL_1$	261.886	232.808	246.687	**224.897**	330.828	285.620	277.014	366.557
		MRL_1	181.525	161.370	170.991	**155.887**	229.312	197.976	192.011	254.078
	0.003	ARL_1	166.037	134.441	148.641	**126.808**	272.936	196.951	185.005	359.980
		$SDRL_1$	165.536	133.940	148.140	**126.307**	272.436	196.450	184.504	359.480
		MRL_1	114.741	92.8404	102.683	**87.5496**	188.838	136.169	127.889	249.172
	0.005	ARL_1	121.569	94.6143	106.469	**88.4215**	231.959	150.276	138.904	353.066
		$SDRL_1$	121.068	94.1130	105.968	**87.9201**	231.458	149.775	138.403	352.566
		MRL_1	83.9182	65.2345	73.4516	**60.9419**	160.435	103.816	95.9339	244.380
	0.01	ARL_1	73.0109	54.6261	62.5781	**50.6150**	168.465	94.5757	85.8392	336.466
		$SDRL_1$	72.5092	54.1238	62.0761	**50.1125**	167.964	94.0744	85.3377	335.966
		MRL_1	50.2599	37.5163	43.0283	**34.7359**	116.424	65.2077	59.1519	232.874
0.1	0.03	ARL_1	28.5195	20.8252	24.2043	**19.2167**	79.8306	38.5752	34.5111	278.779
		$SDRL_1$	28.0150	20.3191	23.6990	**18.7100**	79.3290	38.0719	34.0074	278.279
		MRL_1	19.4196	14.0855	16.4281	**12.9703**	54.9871	26.3902	23.5730	192.888
	0.05	ARL_1	17.9747	13.1761	15.3511	**12.1806**	51.9441	24.5285	21.9395	232.601
		$SDRL_1$	17.4676	12.6662	14.8427	**11.6699**	51.4417	24.0233	21.4337	232.100
		MRL_1	12.1092	8.78184	10.2901	**8.09143**	35.6572	16.6529	14.8580	160.880
	0.10	ARL_1	9.63316	7.21435	8.40412	**6.71331**	27.3556	13.1878	11.8729	152.132
		$SDRL_1$	9.11946	6.69571	7.88829	**6.19316**	26.8509	12.6779	11.3619	151.631
		MRL_1	6.32429	4.64542	5.47140	**4.29743**	18.6127	8.78996	7.87801	105.103
	0.3	ARL_1	3.86580	3.12264	8.40412	**2.96620**	9.13880	5.20609	4.81910	38.5045
		$SDRL_1$	3.32845	2.57454	7.88829	**2.41498**	8.62432	4.67945	4.29006	38.0012
		MRL_1	2.31573	1.79563	5.47140	**1.68576**	5.98127	3.24970	2.98035	26.3412
	0.50	ARL_1	2.70077	2.28935	2.61336	**2.20146**	5.49755	3.56512	3.36322	14.2462
		$SDRL_1$	2.14322	1.71807	2.05336	**1.62634**	4.97247	3.02406	2.81922	13.7371
		MRL_1	1.49884	1.20730	1.43712	**1.14459**	3.45245	2.10560	1.96429	9.52394
	EARL		68.75308	56.52739	62.57035	**53.61388**	118.14096	81.53034	76.60123	213.61487
	ESDRL		68.23093	56.00011	62.04976	**53.08514**	117.63078	81.01440	76.08420	213.10884
	EMRL		47.29918	38.82260	43.01436	**36.80243**	81.53779	56.15858	52.74148	147.71718

Table 5. ARL comparison for the $SAR(1)_{12}$ process on one-sided and two-sided EWMA, MEWMA, and DMEWMA control charts with $\lambda_1 = 0.2, \lambda_2 = 0.3,\ \phi_1 = 0.988,\ \beta = 52.4797$, and $ARL_0 = 370$

a	δ	Control Chart	DMEWMA				MEWMA			EWMA
			f_1= 2.5, f_2=0.5	f_1= 2.5, f_2= 2.5	f_1= 5, f_2= 2.5	f_1= 5, f_2= 5	f_1= 0.5	f_1 = 2.5	f_1 = 5	f_1 = 0
			b= 48.9533	b = 62.4175	b = 91.541	b = 70.485	q= 13.38925	q= 44.6935	q = 61.333	h= 4.2521
	0.000	ARL_0	370.084	370.578	370.266	370.087	370.440	370.514	370.181	370.421
		$SDRL_0$	369.584	370.078	369.766	369.587	369.940	370.014	369.681	369.921
		MRL_0	256.176	256.518	256.302	256.178	256.423	256.474	256.243	256.410
	0.001	ARL_1	364.676	362.913	361.994	**359.751**	364.641	364.316	363.255	365.133
		$SDRL_1$	364.176	362.413	361.494	359.251	364.141	363.816	362.755	364.633
		MRL_1	252.427	251.205	250.568	249.014	252.403	252.178	251.442	252.744
	0.003	ARL_1	354.321	348.497	346.514	**340.723**	353.573	352.522	350.155	354.998
		$SDRL_1$	353.821	347.997	346.014	340.223	353.073	352.022	349.655	354.498
		MRL_1	245.250	241.213	239.838	235.824	244.731	244.003	242.362	245.719
	0.005	ARL_1	344.540	335.186	332.306	**323.610**	343.159	341.470	337.968	345.411
		$SDRL_1$	344.040	334.686	331.806	323.110	342.659	340.970	337.468	344.911
		MRL_1	238.470	231.986	229.990	223.963	237.513	236.342	233.915	239.074
	0.01	ARL_1	322.304	305.976	301.419	**287.523**	319.629	316.657	310.925	323.572
		$SDRL_1$	321.804	305.476	300.919	287.023	319.129	316.157	310.425	323.072
0		MRL_1	223.057	211.740	208.581	198.949	221.203	219.143	215.170	223.936
	0.03	ARL_1	256.221	226.958	219.799	**198.919**	250.887	245.405	235.599	258.305
		$SDRL_1$	255.721	226.457	219.298	198.418	250.387	244.904	235.098	257.805
		MRL_1	177.252	156.968	152.006	137.533	173.555	169.755	162.958	178.697
	0.05	ARL_1	212.683	180.444	173.035	**152.140**	206.538	200.391	189.719	215.001
		$SDRL_1$	212.182	179.943	172.534	151.639	206.037	199.890	189.218	214.500
		MRL_1	147.074	124.727	119.592	105.108	142.814	138.554	131.156	148.680
	0.10	ARL_1	149.392	119.445	113.089	**95.9520**	143.363	137.516	127.729	151.617
		$SDRL_1$	148.891	118.944	112.588	95.4507	142.862	137.015	127.228	151.116
		MRL_1	103.204	82.4459	78.0402	66.1617	99.0247	94.9718	88.1880	104.746
	0.3	ARL_1	68.5644	51.1194	47.7369	**39.0577**	64.8317	61.3381	55.7151	69.9159
		$SDRL_1$	68.0626	50.6169	47.2343	38.5545	64.3298	60.8360	55.2128	69.4141
		MRL_1	47.1778	35.0856	32.7409	26.7247	44.5904	42.1688	38.2711	48.1146
	0.50	ARL_1	44.7328	32.7364	30.4679	**24.7231**	42.1233	39.7072	35.8546	45.6630
		$SDRL_1$	44.2300	32.2325	29.9637	24.2179	41.6203	39.2040	35.3511	45.1602
		MRL_1	30.6585	22.3428	20.7702	16.7878	28.8497	27.1749	24.5043	31.3033
	EARL		214.16654	198.07740	194.26137	**183.55306**	211.14918	208.07103	202.61515	215.43550
	ESDRL		213.66521	197.57565	193.75950	**183.05080**	210.64778	207.56956	202.11354	214.93419
	EMRL		148.10175	136.94941	134.30429	**126.88167**	146.01024	143.87659	140.09480	148.98133
			b = 48.9545	b= 62.4184	b= 91.54145	b= 70.48607	q = 13.39037	q = 44.6947	q = 61.33435	h= 4.2532
	0.000	ARL_0	370.266	370.243	370.955	370.195	370.924	370.962	370.902	370.914
		$SDRL_0$	369.766	369.743	370.455	369.695	370.424	370.462	370.402	370.414
		MRL_0	256.302	256.286	256.780	256.253	256.758	256.785	256.743	256.751
	0.001	ARL_1	364.853	362.591	362.652	**359.853**	365.110	364.749	363.949	365.612
		$SDRL_1$	364.353	362.091	362.152	**359.353**	364.610	364.249	363.449	365.112
		MRL_1	252.550	250.982	251.024	**249.084**	252.728	252.478	251.923	253.076
	0.003	ARL_1	354.489	348.201	347.117	**340.814**	354.014	352.927	350.800	355.449
		$SDRL_1$	353.989	347.701	346.617	**340.314**	353.514	352.427	350.300	354.949
		MRL_1	245.366	241.008	240.256	**235.888**	245.037	244.284	242.809	246.032
	0.005	ARL_1	344.698	334.911	332.860	**323.693**	343.574	341.850	338.569	345.838
		$SDRL_1$	344.198	334.411	332.360	**323.193**	343.074	341.350	338.069	345.338
		MRL_1	238.580	231.796	230.374	**224.020**	237.801	236.606	234.331	239.370
	0.01	ARL_1	322.442	305.748	301.874	**287.588**	319.988	316.984	311.433	323.945
		$SDRL_1$	321.942	305.248	301.374	**287.088**	319.488	316.484	310.933	323.445
		MRL_1	223.153	211.582	208.896	**198.994**	221.452	219.370	215.522	224.195
0.1	0.03	ARL_1	256.308	226.832	220.041	**198.950**	251.107	245.601	235.890	258.539
		$SDRL_1$	255.808	226.331	219.540	**198.449**	250.607	245.100	235.389	258.039
		MRL_1	177.312	156.881	152.174	**137.555**	173.707	169.891	163.160	178.859
	0.05	ARL_1	212.742	180.365	173.185	**152.159**	206.687	200.521	189.908	215.160
		$SDRL_1$	212.241	179.864	172.684	**151.658**	206.186	200.020	189.407	214.659
		MRL_1	147.115	124.673	119.696	**105.122**	142.9177	138.644	131.287	148.791
	0.10	ARL_1	149.421	119.411	113.153	**95.9591**	143.4330	137.577	127.814	151.693
		$SDRL_1$	148.920	118.910	112.652	**95.4578**	142.9321	137.076	127.313	151.192
		MRL_1	103.224	82.4223	78.0846	**66.1666**	99.0732	95.0141	88.2469	104.799
	0.3	ARL_1	68.5703	51.1131	47.7481	**39.0589**	64.8453	61.3499	55.7310	69.9294
		$SDRL_1$	68.0685	50.6106	47.2455	**38.5557**	64.3434	60.8478	55.2287	69.4276
		MRL_1	47.1819	35.0812	32.7487	**26.7255**	44.5999	42.1770	38.2822	48.1240
	0.50	ARL_1	44.7352	32.7338	30.4724	**24.7235**	42.1286	39.7120	35.8611	45.6677
		$SDRL_1$	44.2324	32.2299	29.9682	**24.2183**	41.6256	39.2088	35.3576	45.1649
		MRL_1	30.6602	22.3410	20.7734	**16.7881**	28.8534	27.1782	24.5088	31.3066
	EARL		214.24904	197.94043	194.53567	**183.59305**	211.36350	208.26602	202.91887	215.65728
	ESDRL		213.74771	197.43868	194.03380	**183.09079**	210.86210	207.76455	202.41726	215.15597
	EMRL		148.15893	136.85447	134.49442	**126.90939**	146.15879	144.01175	140.30532	149.13506



(a)

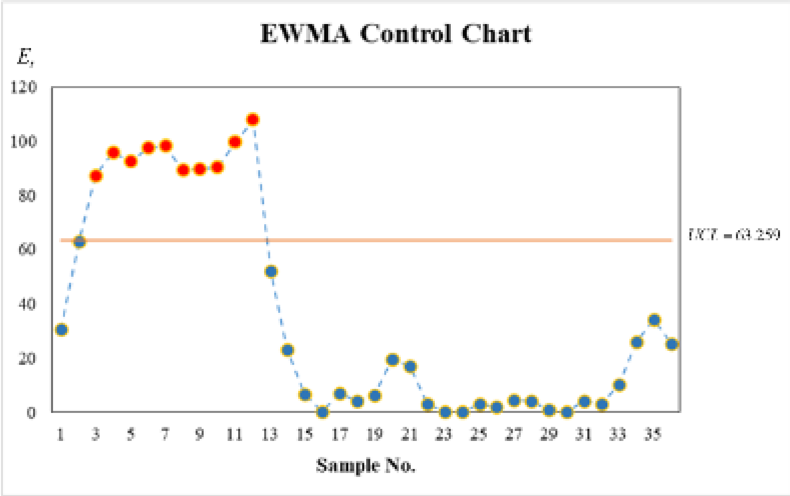


(b)

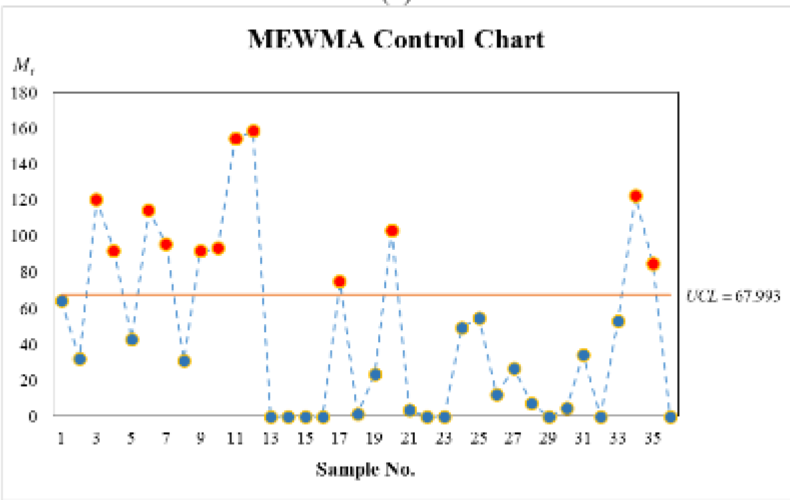
Fig. 1: Comparison of the EARL, ESDRL, and EMRL values for the production index for Biofuel Gasohol 91data when (a) one-sided and (b) two-sided

Table 6. The coefficients for the $SAR(1)_{12}$ model of the production index for Biofuel Gasohol 91 dataset.

model parameters	**$SAR(1)_{12}$**	SE	p-value
SAR	0.988	0.009	0.000
Normalized BIC	8.304		
Residual	**Residual of $SAR(1)_{12}$ model**		
Exponential parameter	52.4797		
One-sample Kolmogorov-Smirnov test	1.179		
p-value	0.124		



(a)



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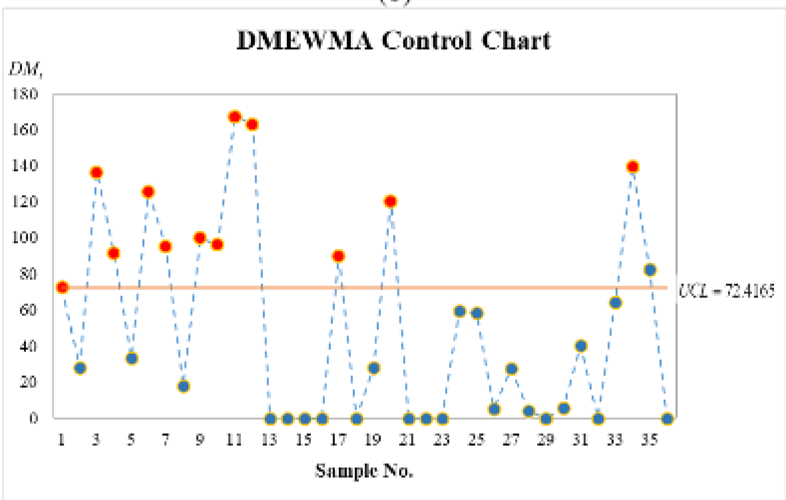

(c)

Fig. 2: The production index for Biofuel Gasohol 91 data with $\lambda_1 = 0.5$ and $\lambda_2 = 0.7$ when (a) EWMA, (b) MEWMA, and (c) DMEWMA control charts