

First-passage time distributions of a Brownian motion observed after a random delay

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Abstract: Suppose that a Brownian motion can only be observed after a random time τ . We are interested in the distribution of the time T required to record the first visit to a given threshold. The cases when τ has an exponential and an Erlang distribution are considered. A method to approximate the probability density function of T from its moments is also presented.

Key-Words: Wiener process, Erlang distribution, convolution, moments, orthogonal polynomials, Laguerre polynomials.

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1 Introduction

Let $\{W(t), t \geq 0\}$ be a Brownian motion (or Wiener process) starting at $W(0) = 0$, with drift parameter $\mu > 0$ and variance parameter σ^2 . The probability density function (p.d.f.) of the first-passage time

$$T_d := \inf\{t > 0 : W(t) = d > 0\} \quad (1)$$

is well known (see, for instance, [1] or, [2]) to be given by

$$f_{T_d}(t) = \frac{d}{\sqrt{2\pi\sigma^2 t^3}} \exp\left\{-\frac{(d - \mu t)^2}{2\sigma^2 t}\right\} \quad (2)$$

for $t > 0$. This density function is a particular *inverse Gaussian distribution* (see, [3]).

Now, suppose that the stochastic process can only be observed after a continuous random time τ , and let T be the time required to record the first visit to the threshold d , so that (assuming that the process is equal to 0 when we begin to observe it)

$$T := \tau + T_d. \quad (3)$$

Moreover, if we assume that the random variables τ and T_d are independent, then we can write that the p.d.f. of T is the convolution product of the p.d.f. of τ with that of T_d :

$$\begin{aligned} f_T(t) &= f_\tau(t) * f_{T_d}(t) := \int_0^t f_\tau(s) * f_{T_d}(t-s) ds \\ &= \int_0^t f_\tau(t-s) * f_{T_d}(s) ds. \end{aligned} \quad (4)$$

In, [4] the case when τ has an exponential distribution with parameter λ was treated:

$$f_\tau(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0. \quad (5)$$

Two cases were considered separately: $\mu^2 \geq 2\lambda\sigma^2$ and $\mu^2 < 2\lambda\sigma^2$. In the first case, it was shown that the p.d.f. of T is given by

$$\begin{aligned} f_T(t) &= \lambda e^{-\lambda t + (d\mu/\sigma^2)} \left[e^{-\kappa d/\sigma^2} \Phi\left(\frac{\kappa t - d}{\sigma\sqrt{t}}\right) \right. \\ &\quad \left. + e^{\kappa d/\sigma^2} \Phi\left(\frac{-\kappa t - d}{\sigma\sqrt{t}}\right) \right] \end{aligned} \quad (6)$$

for $t > 0$, where

$$\kappa := \sqrt{\mu^2 - 2\lambda\sigma^2} \quad (7)$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution $N(0, 1)$.

In the case when $\mu^2 < 2\lambda\sigma^2$, the following expression was obtained:

$$\begin{aligned} f_T(t) &= \lambda \exp\left\{-\frac{(d - \mu t)^2}{2\sigma^2 t}\right\} \\ &\quad \times \Re\left[v\left(\frac{\kappa'\sqrt{t}}{\sqrt{2}\sigma} + i\frac{d}{\sigma\sqrt{2t}}\right)\right] \end{aligned} \quad (8)$$

for $t > 0$, where $\Re$ stands for the *real part*,

$$\kappa' := \sqrt{2\lambda\sigma^2 - \mu^2} \quad (9)$$

and

$$v(z) := e^{-z^2} [1 - \operatorname{erf}(-iz)], \quad (10)$$

in which $\text{erf}(\cdot)$ is the *error function* defined by

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \quad (11)$$

Actually, the function defined in Eq. (6) can be used for any value of the constant κ . Indeed, even if κ becomes an imaginary number (equal to $i\kappa'$) when $\mu^2 < 2\lambda\sigma^2$, the function $f_T(t)$ is still real. For instance, suppose that $\mu = \sigma = \lambda = 1$, so that $\kappa = i$, and $d = 1$. Then the function $f_T(t)$ becomes

$$f_T(t) = e^{1-t} \left[e^{-i} \Phi \left(\frac{it-1}{\sqrt{t}} \right) + e^i \Phi \left(\frac{-it-1}{\sqrt{t}} \right) \right] \quad (12)$$

for $t > 0$. This function is shown in Figure 1.

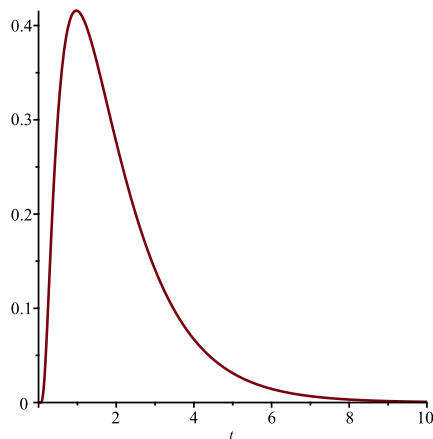


Figure 1: Function $f_T(t)$ defined in Eq. (6) when $\mu = \sigma = \lambda = d = 1$. Source: created by the authors.

In the next section, we will treat the case when τ has an Erlang distribution. Then, in Section 3, we will show how to approximate the p.d.f. of T using its moments and orthogonal polynomials. We will conclude this paper with a few remarks in Section 4.

2 The case when τ has an Erlang distribution

We now assume that τ has an Erlang distribution with parameters $k \in \mathbb{N}$ and λ :

$$f_\tau(t) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t} \quad \text{for } t \geq 0, \quad (13)$$

which generalizes the exponential distribution. The Erlang distribution is a particular gamma distribution, and it is used extensively in queueing theory (see, [5]).

The p.d.f. of T can be expressed as follows:

$$f_T(t) = \frac{d\lambda^k}{\sqrt{2\pi\sigma^2}(k-1)!} e^{-\lambda t} \int_0^t \frac{(t-s)^{k-1}}{s^{3/2}} e^{\lambda s} \times \exp \left\{ -\frac{(d-\mu s)^2}{2\sigma^2 s} \right\} ds. \quad (14)$$

Making use of the *binomial theorem* (see, [6]), we can write that

$$f_T(t) = \frac{d\lambda^k}{\sqrt{2\pi\sigma^2}(k-1)!} \exp \left\{ \frac{d\mu}{\sigma^2} - \lambda t \right\} \times \sum_{n=0}^{k-1} (-1)^n \binom{k-1}{n} t^{k-1-n} I_n(t), \quad (15)$$

where

$$I_n(t) := \int_0^t s^{n-(3/2)} \exp \left\{ -\frac{d^2}{2\sigma^2 s} \right\} \times \exp \left\{ -\frac{\kappa^2 s}{2\sigma^2} \right\} ds. \quad (16)$$

We have

$$I_0(t) = \int_0^t s^{-3/2} \exp \left\{ -\frac{d^2}{2\sigma^2 s} \right\} \times \exp \left\{ -\frac{\kappa^2 s}{2\sigma^2} \right\} ds \\ \stackrel{u=s^{-1/2}}{=} 2 \int_{t^{-1/2}}^\infty \exp \left\{ -\frac{d^2 u^2}{2\sigma^2} \right\} \times \exp \left\{ -\frac{\kappa^2}{2\sigma^2 u^2} \right\} du \quad (17)$$

and the following identity (see, [7], 7.4.33):

$$\int \exp \left\{ -a^2 x^2 - \frac{b^2}{x^2} \right\} dx = \frac{\sqrt{\pi}}{4a} \left[e^{2ab} \times \text{erf} \left(ax + \frac{b}{x} \right) + e^{-2ab} \text{erf} \left(ax - \frac{b}{x} \right) \right] \quad (18)$$

for $a \neq 0$. Moreover, for any $z \in \mathbb{R}$,

$$\text{erf}(z) = 2\Phi(\sqrt{2}z) - 1. \quad (19)$$

It follows that

$$I_0(t) = \frac{\sqrt{\pi}}{\beta} \left[e^{2\alpha\beta} \Phi \left(-\frac{\kappa t + d}{\sigma\sqrt{t}} \right) + e^{-2\alpha\beta} \Phi \left(\frac{\kappa t - d}{\sigma\sqrt{t}} \right) \right], \quad (20)$$

where

$$\alpha := \frac{\kappa}{\sqrt{2}\sigma} \quad \text{and} \quad \beta := \frac{d}{\sqrt{2}\sigma}. \quad (21)$$

Next,

$$\begin{aligned} I_1(t) &= \int_0^t s^{-1/2} \exp \left\{ -\frac{d^2}{2\sigma^2 s} \right\} \\ &\quad \times \exp \left\{ -\frac{\kappa^2 s}{2\sigma^2} \right\} ds \\ &\stackrel{u=s^{1/2}}{=} 2 \int_0^{t^{1/2}} \exp \left\{ -\frac{d^2}{2\sigma^2 u^2} \right\} \\ &\quad \times \exp \left\{ -\frac{\kappa^2 u^2}{2\sigma^2} \right\} du. \end{aligned} \quad (22)$$

Proceeding as above, we obtain that

$$\begin{aligned} I_1(t) &= \frac{\sqrt{\pi}}{\alpha} \left[-e^{2\alpha\beta} \Phi \left(-\frac{\kappa t + d}{\sigma\sqrt{t}} \right) \right. \\ &\quad \left. + e^{-2\alpha\beta} \Phi \left(\frac{\kappa t - d}{\sigma\sqrt{t}} \right) \right] \end{aligned} \quad (23)$$

if $\kappa \neq 0$.

Finally, we can show that the following recurrence relation of second order holds for $n \geq 1$:

$$\begin{aligned} I_{n+1}(t) &= \frac{2n-1}{2\alpha^2} I_n(t) + \frac{\beta^2}{\alpha^2} I_{n-1}(t) \\ &\quad - \frac{1}{\alpha^2} t^{n-(1/2)} \exp \left\{ -\alpha^2 t - \frac{\beta^2}{t} \right\}. \end{aligned} \quad (24)$$

2.1 Particular cases

Case I. Suppose first that $k = 2$, $\mu = \sqrt{2}$ and $\sigma = \lambda = d = 1$. We then have $\kappa = 0$ and $I_0(t)$ reduces to

$$I_0(t) = 2^{3/2} \sqrt{\pi} \left[1 - \Phi \left(\frac{1}{\sqrt{t}} \right) \right]. \quad (25)$$

Moreover,

$$I_1(t) = 2 \int_0^{t^{1/2}} \exp \left\{ -\frac{1}{2u^2} \right\} du, \quad (26)$$

which can be rewritten as follows:

$$\begin{aligned} I_1(t) &= \sqrt{2} \left[\sqrt{2t} \exp \left\{ -\frac{1}{2t} \right\} + \sqrt{\pi} \operatorname{erf} \left(\frac{1}{\sqrt{2t}} \right) \right. \\ &\quad \left. - \sqrt{\pi} \right]. \end{aligned} \quad (27)$$

We have

$$f_T(t) = \frac{1}{\sqrt{2\pi}} e^{\sqrt{2}-t} [t I_0(t) - I_1(t)] \quad \text{for } t \geq 0. \quad (28)$$

This function is displayed in Figure 2.

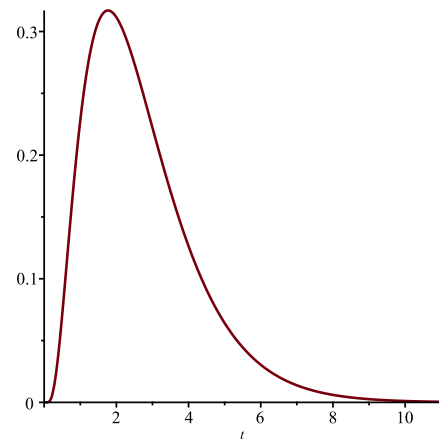


Figure 2: Function $f_T(t)$ defined in Eq. (28) when $k = 2$, $\mu = \sqrt{2}$ and $\sigma = \lambda = d = 1$. Source: created by the authors.

Case II. Assume now that $k = 3$, $\mu = \sigma = \lambda = d = 1$, so that $\kappa = i$, $\alpha = i/\sqrt{2}$ and $\beta = 1/\sqrt{2}$. The functions $I_0(t)$ and $I_1(t)$ are given by

$$I_0(t) = \sqrt{2\pi} \left[e^i \Phi \left(-\frac{it+1}{\sqrt{t}} \right) + e^{-i} \Phi \left(\frac{it-1}{\sqrt{t}} \right) \right] \quad (29)$$

and

$$\begin{aligned} I_1(t) &= i\sqrt{2\pi} \left[e^i \Phi \left(-\frac{it+1}{\sqrt{t}} \right) \right. \\ &\quad \left. - e^{-i} \Phi \left(\frac{it-1}{\sqrt{t}} \right) \right]. \end{aligned} \quad (30)$$

Furthermore, we deduce from Eq. (25) with $n = 1$ that

$$I_2(t) = -I_1(t) - I_0(t) + 2\sqrt{t} \exp \left\{ \frac{t}{2} - \frac{1}{2t} \right\}. \quad (31)$$

The p.d.f. of T is therefore

$$f_T(t) = \frac{1}{2\sqrt{2\pi}} e^{1-t} [t^2 I_0(t) - 2t I_1(t) + I_2(t)] \quad (32)$$

for $t \geq 0$. One can check that $\int_0^\infty f_T(t) dt$ is equal to 1, as required. The function $f_T(t)$ is presented in Figure 3.

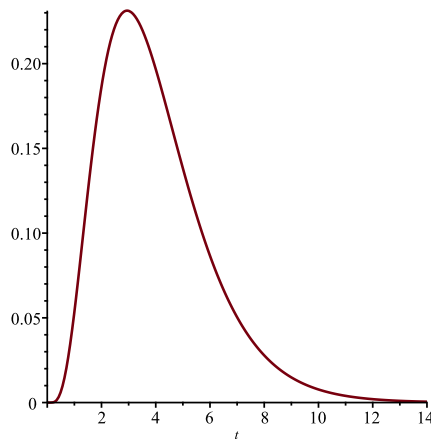


Figure 3: Function $f_T(t)$ defined in Eq. (32) when $k = 3$ and $\mu = \sigma = \lambda = d = 1$. Source: created by the authors.

3 Polynomial approximations

As we have seen in Section 2, obtaining an exact expression (without integrals) for the random variable T can be difficult. Moreover, the expression is in general rather involved. In this section, we will present a method that can be used for any random variable τ , provided that we can compute its characteristic function, and which yields a very good polynomial approximation to the p.d.f. of T .

The characteristic function of the random variable T_d is known to be (for $\mu > 0$)

$$C_{T_d}(\omega) := E[e^{i\omega T_d}] \quad (33)$$

$$= \exp \left\{ \frac{d\mu}{\sigma^2} \left[1 - \left(1 - \frac{2i\sigma^2\omega}{\mu^2} \right)^{1/2} \right] \right\}.$$

Moreover, by independence, we can write that

$$C_T(\omega) = C_\tau(\omega) \cdot C_{T_d}(\omega). \quad (34)$$

Hence, if we have an explicit expression for $C_\tau(\omega)$, we can compute the moments of T by differentiating $C_T(\omega)$.

Next, let $\{L_n(x), n = 0, 1, \dots\}$ be a set of orthonormal polynomials (see, [8]) on the interval $[0, \infty)$ with respect to the weight function $w(x)$. We define

$$u_n(x) = \sqrt{w(x)} L_n(x). \quad (35)$$

Then, any function $g(x)$ can be expressed as follows:

$$g(x) = \sum_{n=0}^{\infty} c_n u_n(x), \quad (36)$$

where

$$c_n := \int_0^{\infty} g(x) u_n(x) dx. \quad (37)$$

Now, if $f_X(x)$ is the p.d.f. of the random variable X , we write (see, [9])

$$f_X(x) = \sqrt{w(x)} \sum_{n=0}^{\infty} c_n u_n(x)$$

$$= w(x) \sum_{n=0}^{\infty} c_n L_n(x). \quad (38)$$

It follows that

$$c_n = \int_0^{\infty} \frac{f_X(x)}{\sqrt{w(x)}} u_n(x) dx = \int_0^{\infty} f_X(x) L_n(x) dx. \quad (39)$$

Hence, the constants c_n depend only on the moments of X .

In this paper, we will use the *Laguerre polynomials* (see, [10]). The corresponding weight function is $w(x) = e^{-x}$. The first five polynomials are given by

$$L_0(x) = 1,$$

$$L_1(x) = -x + 1,$$

$$L_2(x) = \frac{1}{2}(x^2 - 4x + 2),$$

$$L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6),$$

$$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24),$$

$$L_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120).$$

Let us consider the case when τ has an exponential distribution with parameter λ . We have

$$C_\tau(\omega) = \frac{\lambda}{\lambda - i\omega}. \quad (40)$$

With $d = 1$ and the parameters $\mu = \sigma = \lambda = 1$ used to compute the function $f_T(t)$ given in Eq. (12), we find that $E[T] = 2$, $E[T^2] = 6$, $E[T^3] = 25$, $E[T^4] = 137$ and $E[T^5] = 951$. We then calculate $c_0 = 1$, $c_1 = -1$, $c_2 = 0$, $c_3 = -1/6$, $c_4 = 1/24$ and $c_5 = -1/20$.

Let

$$P_m(t) := e^{-t} \sum_{n=0}^m c_n L_n(t). \quad (41)$$

The polynomials $P_4(t)$ and $P_5(t)$ are shown in Figure 4, together with the exact p.d.f. $f_T(t)$ defined in Eq. (12). We see that the polynomial of degree 5

$$P_5(t) = e^{-t} \left(-\frac{7}{40} + \frac{19}{12}t - \frac{3}{8}t^2 + \frac{1}{12}t^3 - \frac{5}{576}t^4 + \frac{1}{2400}t^5 \right) \quad (42)$$

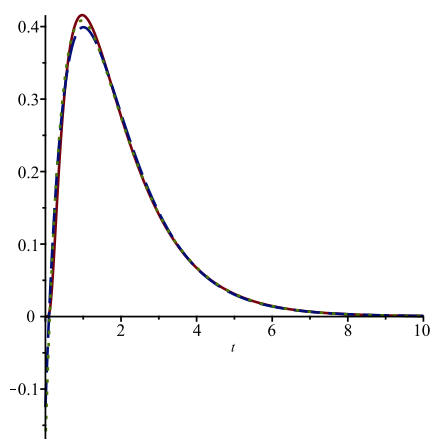


Figure 4: Polynomials $P_4(t)$ (dashed line) and $P_5(t)$ (dotted line), and the exact p.d.f. $f_T(t)$ (solid line) defined in Eq. (12). Source: created by the authors.

is a very good approximation to the function $f_T(t)$. Moreover, the convergence (at least in this case) is fairly rapid. Finally, it is obviously easier to use the polynomial approximation than the exact expression for $f_T(t)$, which involves special functions of complex arguments.

4 Concluding remarks

In this paper, a problem which has applications in various fields has been considered. First, we generalized the results found in, [4]. Then, in Section 3, we showed that it was possible to obtain excellent polynomial approximations of the exact density function.

Following on from this work, we could consider other diffusion processes, such as geometric Brownian motion or the Ornstein-Uhlenbeck process.

Finally, we could also try to treat this problem in discrete time, replacing the diffusion process by a Markov chain, and assuming that the random time delay τ is also a discrete random variable.

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Conflict of Interest

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